

Preorientations of the derived motivic multiplicative group

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Abstract

We provide a proof in the language of model categories and symmetric spectra of Lurie's theorem that topological complex K -theory represents orientations of the derived multiplicative group. Then we generalize this result to the motivic situation. Along the way, a number of useful model structures and Quillen adjunctions both in the classical and in the motivic case are established.

1 Introduction

Starting with the definition of elliptic cohomology via Landweber exactness in the late 80s, the interplay between stable homotopy theory and modular forms has been one of a very active area of research in the last two decades. The construction due to Goerss, Hopkins and Miller of the spectrum of *topological modular forms* (tmf for short) as the universal elliptic E_∞ -ring spectrum is one of the most (maybe *the* most) important achievements in this area. See [Be] for an overview of how the different pieces of the construction of tmf fit together.

Another topic in stable homotopy theory that has become more and more popular in the last couple of years is doing commutative algebra for E_∞ -ring spectra. This can now be done in a nice way thanks to the construction of strict monoidal model categories underlying the stable homotopy category, using e. g. S -modules [EKMM] or symmetric spectra [HSS]. Glueing these “derived” commutative ring objects together leads to one of the possible frameworks for derived algebraic geometry, with classical algebraic geometry embedded via the Eilenberg-Mac Lane functor.

Very recently, Jacob Lurie merged these two areas in a fascinating way. He gave a conceptual definition of tmf as the solution of a moduli problem in derived algebraic geometry. More precisely, tmf are the global sections of a sheaf of E_∞ -ring spectra classifying oriented derived elliptic curves. Lurie has sketched the proof of this theorem in [Lu1], and has lectured about various parts of it at various places. His point of view is that the best

language to state and prove the theorem is the one of infinity categories rather than the one of model categories, and we have no reason to doubt he is right. Here infinity categories really mean quasi-categories, also known as weak Kan complexes, as first invented by Boardman and Vogt [BV] and recently studied in great detail by Lurie, Joyal and others. The interested reader should consult Lurie's homepage and [Lu2], as well as [Ber] for a comparison with other approaches to infinity categories. We expect that Lurie will publish a detailed proof of his theorem in this language in the near future, the book [Lu2] and the preprints of Lurie containing already most of the necessary language and machinery.

The above description of tmf (corresponding to height 2 and the second chromatic layer) has an analog in height 1 which is much easier to state and to prove, and is also due to Lurie [Lu1, section 3]. Namely, real topological K -theory KO classifies oriented derived multiplicative groups. The key step for proving this is to show that the suspension spectrum of \mathbf{CP}^∞ classifies preorientations of the derived multiplicative group. Here the derived multiplicative group is by definition $\mathbf{G}_m := Spec(\Sigma^\infty \mathbf{Z}_+)$, the name being justified by classical algebraic geometry over a base field k , where the multiplicative group is $Spec(k[\mathbf{Z}])$. As usual, the object $Rmap_{AbMon(Sp^\Sigma)}(\Sigma^\infty \mathbf{Z}_+, -)$ it represents via the derived version of the Yoneda embedding will still be called the multiplicative group. (In the present preprint, all arguments take place in the *affine* derived setting, so there is no need to write $Spec$ and to reverse the order of the arrows everywhere.) We will provide a proof of this result in the language of model categories and symmetric spectra. It reads as follows in general, the special case $N = \mathbf{CP}^\infty$ being the one discussed above:

Theorem 1.1 (*Lurie*) *For any abelian monoid A in symmetric spectra Sp^Σ (based on simplicial sets) and any simplicial abelian group N , we have a natural isomorphism of abelian groups*

$$\begin{aligned} & Hom_{Ho(AbMon(Sp^\Sigma))}(\Sigma^\infty N_+, A) \\ & \simeq Hom_{Ho(AbMon(\Delta^{op}Sets))}(N, Rmap_{AbMon(Sp^\Sigma)}(\Sigma^\infty \mathbf{Z}_+, A)) \\ & = Hom_{Ho(AbMon(\Delta^{op}Sets))}(N, \mathbf{G}_m(A)). \end{aligned}$$

Here $Ho(-)$ denotes the homotopy category, $Rmap$ means the derived mapping space and the weak equivalences between abelian monoids are always the underlying ones, forgetting the abelian monoid structure. The model structures involved in this statement are discussed in detail in section

3. Beware that in general the category of abelian monoids in a homotopy category of a monoidal model category is different from the homotopy category of abelian monoids in the monoidal model category, the monoidal model category $(\Delta^{op}Sets, \times)$ and the abelian monoid QS^0 in $Ho(\Delta^{op}Sets)$ being the most prominent example.

We will prove Theorem 1.1 in section 4, and indeed a much more general version (see below). Using a theorem of Snaith [Sn], Lurie's definition of an orientation and his above theorem then imply that complex K -theory represents orientations of the standard derived multiplicative group, and then further that KO represents oriented derived multiplicative groups in general. We refer the reader to section 5 for further details.

The above theorem can be generalized to motivic symmetric spectra $Sp^{\Sigma, T}(\mathcal{M})$ on the smooth Nisnevich site $\mathcal{M} = (Sm/S)_{Nis}$ with S an arbitrary noetherian base scheme as follows, everything equipped with appropriate motivic (that is \mathbf{A}^1 -local) model structures as discussed in section 3, and \mathbf{G}_m defined using the suspension spectrum with respect to a given motivic circle T .

Theorem 1.2 *Let $\mathcal{M} = (Sm/S)_{Nis}$ and $T = S^1$ or $T = \mathbf{P}^1$. Then for any abelian monoid A in motivic symmetric T -spectra $Sp^{\Sigma, T}(\mathcal{M})$ and any abelian group N in the category $\Delta^{op}PrShv(\mathcal{M})$ of simplicial presheaves on \mathcal{M} , we have a natural isomorphism of abelian groups*

$$\begin{aligned} & Hom_{Ho(AbMon(Sp^{\Sigma, T}(\mathcal{M})))}(\Sigma_T^\infty N_+, A) \\ & \simeq Hom_{Ho(AbMon(\Delta^{op}PrShv(\mathcal{M})))}(N, Rmap_{AbMon(Sp^{\Sigma, T}(\mathcal{M}))}(\Sigma_T^\infty \mathbf{Z}_+, A)) \\ & = Hom_{Ho(AbMon(\Delta^{op}PrShv(\mathcal{M})))}(N, \mathbf{G}_m(A)). \end{aligned}$$

This is the main theorem of this preprint. Applying it to $T = \mathbf{P}^1$ pointed at ∞ and to $N = \mathbf{P}^\infty$ which is not a variety but still a simplicial presheaf, and using the recently established motivic version of Snaith's theorem [GS], [SO], it will imply that *algebraic K -theory represents motivic orientations of the derived motivic multiplicative group*, provided one works with the correct motivic generalizations of the concept of derived algebraic groups and of orientations. Again, we refer to section 5 for details, as well as for possible connections to hermitian K -theory which in many ways is the motivic analog of topological real K -theory.

One of the many motivations of this preprint is that the generalizations of the language of derived algebraic geometry from classical to motivic

spectra should ultimately lead to a definition of a motivic version of tmf , generalizing the above Theorem 1.1 of Lurie about height 2 to the motivic set-up as well. We will not pursue this in the present preprint. (Note that the recent article [NSO] allows to define motivic elliptic cohomology theories and motivic elliptic ring spectra via motivic Landweber exactness.) However, concerning motivic derived algebraic geometry, we wish to point out two interesting applications of the results of this preprint. First, motivic symmetric \mathbf{P}^1 -spectra equipped with a suitable positive model structure satisfy the axioms of a HA -context of Toen and Vezzosi [TV2], so their machinery applies to this example. Second, the motivic analogue of the axioms of Goerss and Hopkins [GH] is also satisfied (see Theorem 3.15). I understand that this second application was established simultaneously and independently by Paul Arne Østvær, who wants to use it for doing motivic obstruction theory. Both applications are presented in section 3.

We pause to make some comments concerning the proof of the main theorem, that is Theorem 1.2, which is given in section 4 and uses the results established in section 3. First, one should notice that the theorem is about T -spectra, but even for $T = \mathbf{P}^1$ the proof involves motivic S^1 -spectra as well. This is mainly due to the fact that at some point one needs a motivic version of the recognition principle which relates E_∞ -spaces to connective S^1 -spectra. The classical recognition principle is a statement about S^1 -deloopings, and our generalization of it to motivic S^1 -spectra is sufficient for our purposes. Finding a recognition principle for motivic \mathbf{P}^1 -spectra, that is a motivic operad encoding \mathbf{G}_m - or \mathbf{P}^1 -deloopings, remains one of the main open problems in motivic homotopy theory, as already pointed out by Voevodsky in [Vo2, introduction]. To show that a motivic version of the recognition principle with respect to S^1 holds, a previous version of this preprint was using the beautiful \mathbf{A}^1 -connectivity theorem of Fabien Morel [Mo2], which is known only over a field. (It seems to be an open question if Morel's theorem also holds for 1-dimensional base schemes. For 2-dimensional base schemes, there is a counter-example due to Ayoub.) However, we later realized that the proof does not really require this result and hence holds for more general base schemes.

Keeping in mind these points, the strategy for proving the main theorem may be very roughly described as follows: First write down the proof in the classical case, that is of Theorem 1.1, choosing arguments as abstract and as conceptual as possible. This part is already very interesting on its own right, as so far no complete proof of Lurie's theorem in the language of model categories seems to be available in the literature. Second, show

that this proof generalizes to diagram categories (namely over the Nisnevich site $(Sm/S)_{Nis}$ of smooth varieties over the given base scheme S) and is well behaved under (left) Bousfield localization with respect to Nisnevich descent and to the affine line \mathbf{A}^1 . When stabilizing in the motivic situation, always make the good choice for the circle (both S^1 and \mathbf{P}^1 do appear in the proofs) and for the model structure at every stage. Indeed, it will turn out that during the various proofs we have to consider many different model structures for symmetric spectra and their lifts to modules over rings and operads. Some of these model structures are new, and their existence is of independent interest, so their presentation here should also serve for future reference. (Recall that the first model structures on motivic symmetric spectra are due to Jardine [Ja1] and Hovey [Ho2].) In particular, I am not aware of any discussion of model structures and derived mapping spaces for commutative motivic symmetric ring spectra in the literature so far.

Carrying out the above strategy requires that various Quillen adjunctions and equivalences are stable under localization in a suitable sense, see e. g. [Hi2, Theorem 3.3.20] for a result in this direction. We will provide all details in the parts of the proof concerning classical spectra. When passing to diagram categories and motivic Bousfield localizations of those, we will provide details in the first couple of proofs, but allow ourselves to skip some of the by then familiar arguments in some of the later proofs. The reader interested in the classical case should simply think of the trivial site and ignore all localization functors with respect to the Nisnevich topology or to the affine line \mathbf{A}^1 . The proof then becomes considerably shorter. In particular, most (but not all, see Proposition 3.9) model structures discussed in section 3 are known in that case.

At first glance, it might be surprising that we need E_∞ -structures to prove a theorem about strictly commutative monoids in strictly monoidal model categories. This is essentially a consequence of the Lewis paradoxon, as explained at the end of section 2. If one is only interested in strict adjunctions and willing to ignore all derived information, and in particular to sacrifice homotopy invariance of the statement, then there is a much easier proof not using operads, which we present in section 2. At the beginning of section 2, we fix some notations which will be used throughout this preprint.

This work started as a joint project with Niko Naumann, and was presented as such on a conference in Münster in July 2009. I am indebted to him for the many discussions we had on the topics of this preprint. Some parts of the work presented here have been obtained in joint work or are at least influenced by these discussions, and I thank him for allowing me to

include these parts here. Moreover, I wish to thank Stefan Schwede, John Harper and Benoit Fresse for discussions and explanations about certain points in their works concerning model structures for classical symmetric spectra, operads over them and E_∞ -operads, respectively, as well as Jacob Lurie for some explanations about [Lu1] and Pablo Pelaez for discussions related to [Pe] and [Mo2].

2 The non-derived situation: preorientations which are not homotopy invariant

The main goal of this section is to establish the following theorem, which is a non-derived analogue of Theorems 1.1 and 1.2. All the objects involved are defined below, and the proof of the theorem is given by suitably combining the lemmata in this section.

Theorem 2.1 *Let \mathcal{C} be a category, let M be an abelian group object in $\Delta^{op}PrShv(\mathcal{C})$ and A an abelian monoid object in $Sp^{\Sigma,T}(\mathcal{C})$. Then we have a natural adjunction isomorphism of simplicial sets*

$$map_{AbMonSp^{\Sigma,T}(\mathcal{C})}(S[M], A) \cong map_{AbMon\Delta^{op}PrShv(\mathcal{C})}(M, \mathbf{G}'_m(A)).$$

We now introduce some notation. For any monoidal category $\mathcal{D} = (\mathcal{D}, \otimes)$, we denote the category of monoid objects in \mathcal{D} by $Mon\mathcal{D}$, and the one of abelian monoids by $AbMon\mathcal{D}$. In our applications, \otimes will be either the cartesian product \times or some smash product \wedge . All monoids are assumed to be associative, but not necessarily unital. We refer the reader to [ML] for precise definitions of monoidal categories, (strong) monoidal functors etc.

We fix a category \mathcal{C} from now on. In applications \mathcal{C} will be a site, more specifically either the trivial site or the site $(Sm/k)_{Nis}$ of smooth k -schemes, k a field, with the Nisnevich topology. We denote the category of simplicial presheaves on \mathcal{C} by $\Delta^{op}PrShv(\mathcal{C})$. For a given simplicial presheaf T , we denote the category of presheaves of symmetric T -spectra on \mathcal{C} by $Sp^{\Sigma,T}(\mathcal{C})$. Model structures on these categories are discussed in [MV], [Ja1], [Ho2] and elsewhere, but we won't need them in this subsection. One might wish to call commutative monoids in $Sp^{\Sigma,T}(\mathcal{C})$ “commutative motivic symmetric ring spectra” in case $\mathcal{C} = (Sm/k)_{Nis}$, resp. “commutative symmetric ring spectra” in case $\mathcal{C} = pt$.

Adding a disjoint base point is denoted by $(\)_+$ and yields a left adjoint to the functor F forgetting the base point in various situations. For a simplicial presheaf X , we set $S[X] := \Sigma_T^\infty(X_+) \in Sp^{\Sigma,T}(\mathcal{C})$ which is defined objectwise

as in [HSS, Definition 2.2.5] or [Sc2, Example 1.2.6]. If X is a monoid in the monoidal category $(\Delta^{op}PrShv(\mathcal{C}), \times)$, then X_+ is a monoid in the monoidal category $(\Delta^{op}PrShv(\mathcal{C})_\bullet, \wedge)$ of pointed simplicial presheaves and $S[X]$ is a monoid in the monoidal category $(Sp^{\Sigma, T}(\mathcal{C}), \wedge)$ of presheaves of symmetric T -spectra. We denote the functor sending a presheaf of symmetric T -spectra to the simplicial presheaf sitting in degree 0 by Ev_0 .

For simplicial presheaves \mathcal{F} and \mathcal{G} we have a simplicial set $map_{\Delta^{op}PrShv(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ given by $map_{\Delta^{op}PrShv(\mathcal{C})}(\mathcal{F}, \mathcal{G})_n := Hom_{\Delta^{op}PrShv(\mathcal{C})}(\mathcal{F} \times \Delta^n, \mathcal{G})$. We define the simplicial presheaf $\mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\mathcal{F}, \mathcal{G})$ by $\mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\mathcal{F}, \mathcal{G})(c) = map_{\Delta^{op}(\mathcal{C}/c)}(\mathcal{F}|_c, \mathcal{G}|_c)$ where $F|_c$ denotes the restriction of the (simplicial) presheaf \mathcal{F} to the category \mathcal{C}/c of objects in \mathcal{C} lying over c . For presheaves of symmetric T -spectra, we define the simplicial sets $map_{Sp^{\Sigma, T}(\mathcal{C})}$ and simplicial presheaves $\mathbf{map}_{Sp^{\Sigma, T}(\mathcal{C})}$ in a similar way. Forgetting about simplicial enrichments, we write \mathbf{Hom} for the presheaf version of Hom .

Finally, we define the (non-derived) multiplicative group as follows, the derived version of the introduction being the one using the *derived* mapping space $Rmap$ instead.

Definition 2.2 *The non-derived multiplicative group is the functor*

$$\mathbf{G}'_m : AbMonSp^{\Sigma, T}(\mathcal{C}) \rightarrow AbMon\Delta^{op}PrShv(\mathcal{C})$$

given by

$$\mathbf{G}'_m(A) := \mathbf{map}_{AbMonSp^{\Sigma, T}(\mathcal{C})}(S[\mathbf{Z}], A)$$

where $\mathbf{map}_{AbMonSp^{\Sigma, T}(\mathcal{C})}$ is introduced in Definition 2.5 below. The monoid structure on $\mathbf{G}'_m(A)$ is induced by the comonoid structure on $S[\mathbf{Z}]$.

We define monoidal and strict monoidal functors between monoidal categories and monoidal transformations between (strong) monoidal functors as in [ML, chapter XI]. An adjunction between monoidal categories is called a *monoidal adjunction* if the unit and the counit are monoidal transformations. One easily checks that a monoidal functor sends monoids to monoids.

Lemma 2.3 (i) *We have a monoidal adjunction*

$$(\)_+ : \Delta^{op}PrShv(\mathcal{C}) \xrightarrow{\sim} \Delta^{op}PrShv(\mathcal{C})_\bullet : F$$

where $(\)_+$ is strong monoidal and the forgetful functor F is monoidal. Consequently, we have isomorphisms

$$Hom_{Mon\Delta^{op}PrShv(\mathcal{C})}(M, F(N)) \simeq Hom_{Mon\Delta^{op}PrShv(\mathcal{C})_\bullet}(M_+, N)$$

for any unpointed monoid $M \in \text{Mon}\Delta^{\text{op}}\text{PrShv}(\mathcal{C})$ and any pointed monoid $N \in \text{Mon}\Delta^{\text{op}}\text{PrShv}(\mathcal{C})_{\bullet}$.

(ii) We have a monoidal adjunction

$$\Sigma_T^{\infty} : \Delta^{\text{op}}\text{PrShv}(\mathcal{C})_{\bullet} \xrightarrow{\sim} \text{Sp}^{\Sigma, T}(\mathcal{C}) : \text{Ev}_0$$

where both functors are strong monoidal. Consequently, we have isomorphisms

$$\text{Hom}_{\text{Mon}\Delta^{\text{op}}\text{PrShv}(\mathcal{C})_{\bullet}}(A, \text{Ev}_0(B)) \simeq \text{Hom}_{\text{Mon}\text{Sp}^{\Sigma, T}(\mathcal{C})}(\Sigma_T^{\infty} A, B)$$

for any monoid A in $\Delta^{\text{op}}\text{PrShv}(\mathcal{C})_{\bullet}$ and any monoid B in $\text{Sp}^{\Sigma, T}(\mathcal{C})$.

(iii) We have a monoidal adjunction

$$S[\] : \Delta^{\text{op}}\text{PrShv}(\mathcal{C}) \xrightarrow{\sim} \text{Sp}^{\Sigma, T}(\mathcal{C}) : F \circ \text{Ev}_0$$

where $S[\]$ is strong monoidal and $F \circ \text{Ev}_0$ is monoidal. Consequently, we have isomorphisms

$$\text{Hom}_{\text{Mon}\Delta^{\text{op}}\text{PrShv}(\mathcal{C})}(M, F \circ \text{Ev}_0(B)) \simeq \text{Hom}_{\text{Mon}\text{Sp}^{\Sigma, T}(\mathcal{C})}(S[M], B)$$

for any monoid M in $\Delta^{\text{op}}\text{PrShv}(\mathcal{C})$ and any monoid B in $\text{Sp}^{\Sigma, T}(\mathcal{C})$.

Proof: Part (i) is straightforward. The morphisms of [ML, XI.2.(1),(2)] for the monoidal functor F are given by the quotient map $X \times Y \rightarrow X \wedge Y$ and by $pt \rightarrow (pt)_{+} = S^0$. The final statement follows from the obvious remark that a monoidal adjunction induces an adjunction between categories of monoids.

Part (ii) is checked objectwise, using [HSS, Definition 2.2.5, Proposition 2.2.6.1, Definition 2.1.3] or the corresponding results in [Sc2] and then proceeding similar to part (i).

Part (iii) is obtained by composing (i) and (ii). \square

Definition 2.4 Let (\mathcal{D}, \otimes) be a monoidal category such that the underlying category is enriched over simplicial sets. We say that (\mathcal{D}, \otimes) satisfies (MS) if there is a natural transformation of simplicial sets $\tau_{x,y,z,w} : \text{map}(x, y) \times \text{map}(z, w) \rightarrow \text{map}(x \otimes z, y \otimes w)$ which on $\text{map}(_, _)_0 = \text{Hom}$ coincides with the transformation sending (f, g) to $f \otimes g$, and we say that $(\mathcal{D}, \otimes, \tau)$ is a simplicial monoidal category.

Definition 2.5 Let $(\mathcal{D}, \otimes, \tau)$ be a simplicial monoidal category. Then for any monoids (x, m_x) and (y, m_y) in \mathcal{D} , we define $\text{map}_{\text{Mon}}(x, y) \subset \text{map}(x, y)$ to be the equalizer of $\text{map}(m_x, y) : \text{map}(x, y) \rightarrow \text{map}(x \otimes x, y)$ and $\text{map}(x \otimes x, m_y) \circ \tau_{x,y,x,y} \circ \Delta : \text{map}(x, y) \rightarrow \text{map}(x \otimes x, y)$ where $\tau_{x,y,x,y}$ is as in Definition 2.4. If $\mathcal{D} = \Delta^{\text{op}}\text{PrShv}(\mathcal{C})$, $\mathcal{D} = \Delta^{\text{op}}\text{PrShv}(\mathcal{C})_{\bullet}$ or $\mathcal{D} = \text{Sp}^{\Sigma, T}(\mathcal{C})$, then we denote the presheaf version of map_{Mon} by $\mathbf{map}_{\text{Mon}}$, and the one of Hom_{Mon} by $\mathbf{Hom}_{\text{Mon}}$.

Recall that we consider monoids without units. Considering mapping spaces $\text{map}_{\text{Mon}^{\text{unital}}}$ between unital monoids would lead to the cartesian diagram

$$\begin{array}{ccc} \text{map}_{\text{Mon}^{\text{unital}}}(x, y) & \xrightarrow{\quad} & \Delta^0 \\ \downarrow & & \downarrow u_y \\ X(x, y) & \xrightarrow{\text{map}(u_x, \text{id}_y)} & \text{map}(1_{\mathcal{D}}, y) \end{array}$$

where u_x and u_y are the units of x and y .

Lemma 2.6 Lemma 2.3 above remains true when replacing Hom by \mathbf{Hom} or by \mathbf{map} everywhere.

Proof: The isomorphisms for Hom_{Mon} formally imply those for $\mathbf{Hom}_{\text{Mon}}$. The claim about $\mathbf{map}_{\text{Mon}}$ follows from Lemma 2.8 below. \square

Definition 2.7 Let $(\mathcal{C}, \otimes, \tau)$ and $(\mathcal{D}, \otimes, \tau)$ be simplicial monoidal categories as in Definition 2.4. Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor of simplicial categories such that the underlying functor of categories is a monoidal functor with structure maps $F_2 : F(x) \otimes F(y) \rightarrow F(x \otimes y)$ and $F_0 : 1_{\mathcal{D}} \rightarrow F(1_{\mathcal{C}})$. We say that F is a simplicial monoidal functor if for any objects x, y of \mathcal{C} the diagram

$$\begin{array}{ccc} \text{map}(Fx, Fy) \times \text{map}(Fx, Fy) & \xrightarrow{F\tau_{\mathcal{C}}} & \text{map}(F(x \otimes x), F(y \otimes y)) \\ \downarrow \tau_{\mathcal{D}} & & \downarrow F_2^* \\ \text{map}(Fx \otimes Fx, Fy \otimes Fy) & \xrightarrow{F_{2*}} & \text{map}(Fx \otimes Fx, F(y \otimes y)) \end{array}$$

commutes.

Lemma 2.8 (i) Assume that $(\mathcal{D}_i, \otimes_i, \tau_i)$, $i = 1, 2$ are simplicial monoidal categories and that

$$\alpha : \mathcal{D}_1 \xleftarrow{\quad} \mathcal{D}_2 : \beta$$

is a simplicial monoidal adjunction, i. e. α and β are simplicial monoidal functors and there is a monoidal adjunction between the underlying monoidal functors.

Then for (x_i, m_i) monoids in \mathcal{D}_i , $i = 1, 2$, we have an isomorphism

$$\mathrm{map}_{\mathrm{Mon}\mathcal{D}_1}(x_1, \beta x_2) \simeq \mathrm{map}_{\mathrm{Mon}\mathcal{D}_2}(\alpha x_1, x_2)$$

of simplicial sets, and similarly for **map**.

(ii) The monoidal categories $(\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C}), \times), (\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})_{\bullet}, \wedge)$ and $(\mathrm{Sp}^{\Sigma, T}(\mathcal{C}), \wedge)$ are enriched over simplicial sets as categories and satisfy (MS) with respect to the obvious choices of τ , hence are simplicial monoidal categories.

(iii) The monoidal adjunctions of Lemma 2.3 are simplicial.

Proof: The proof of (i) is a little long but again straightforward. In part (ii), for constructing the transformations τ required in (MS) one uses the diagonal $\Delta_+^n \rightarrow \Delta_+^n \wedge \Delta_+^n$, the twist and that for any simplicial presheaf K (in particular for $K = \Delta^n$) and any $X \in \mathrm{Sp}^{\Sigma, T}(\mathcal{C})$ one has $K \wedge X = (\Sigma_T^\infty K) \wedge X$ which can be shown objectwise using the results of [Sc2, Chapter I]. For part (iii), use that all mapping spaces involved are defined using the standard cosimplicial object, and the composition is defined using the diagonal on it.

□

Observe that for any monoidal category \mathcal{C} , one has $\mathrm{Hom}_{\mathrm{Mon}\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathrm{AbMon}\mathcal{C}}(A, B)$ for any abelian monoid objects A and B in \mathcal{C} , and similar for **Hom**, **map** and **map**. E. g., both Lemma 2.3 and Lemma 2.9 below restrict to *abelian* monoids. For $M \in \mathrm{Mon}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})$, we denote by M^\times the group object in $\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})$ defined by $(M^\times)_k = (M_k)^\times$, that is taking objectwise the invertible elements in each simplicial degree. These units satisfy the following.

Lemma 2.9 (i) For any $N \in \mathrm{Mon}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})$ a simplicially constant group object and $M \in \mathrm{Mon}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})$, one has an isomorphism

$$\mathrm{map}_{\mathrm{Mon}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})}(N, M) \simeq \mathrm{map}_{\mathrm{Groups}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})}(N, M^\times)$$

and similar for **map**. In particular, if $N = \mathbf{Z}$ one has

$$\mathbf{map}_{\mathrm{Mon}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})}(\mathbf{Z}, M) \simeq M^\times.$$

(ii) More generally, if N is a group object in $\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})$, then one has an isomorphism

$$\mathrm{map}_{\mathrm{Mon}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})}(N, M) \simeq \mathrm{map}_{\mathrm{Groups}\Delta^{\mathrm{op}}\mathrm{PrShv}(\mathcal{C})}(N, M^\times)$$

and similarly for **map**.

Proof: For simplicially constant N , one has isomorphisms

$map_{Mon\Delta^{op}PrShv(\mathcal{C})}(N, M)_n \simeq Hom_{Mon\Delta^{op}PrShv(\mathcal{C})}(N, \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M))$
 $\simeq Hom_{MonPrShv(\mathcal{C})}(N, \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M)_0) \simeq Hom_{MonPrShv(\mathcal{C})}(N, M_n)$,
 where the first isomorphism holds because $(\)_n$ commutes with limits. The monoid structure on $\mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M)$ is defined composing τ , the diagonal $\Delta^n \rightarrow \Delta^n \times \Delta^n$ and the monoid structure of M . Using these isomorphisms, part (i) about map reduces to the corresponding well-known result for usual monoids, and the result about \mathbf{map} follows formally from this by definition. For the claim about \mathbf{Z} use that $\mathbf{map}_{Mon\Delta^{op}PrShv(\mathcal{C})}(\mathbf{Z}, M)_n \simeq \mathbf{Hom}_{Mon}(\mathbf{Z}, \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M))$ and that $Hom_{Mon}(\mathbf{Z}, M) = M^\times$ for usual monoids M . For (ii), one first checks that $\mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n \times \Delta^k, \mathbf{map}_{Mon\Delta^{op}PrShv(\mathcal{C})}(\mathbf{Z}, M)) \simeq \mathbf{map}_{Mon\Delta^{op}PrShv(\mathcal{C})}(\mathbf{Z}, \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n \times \Delta^k, M))$ as both are subsets of $\mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n \times \Delta^k \times \mathbf{Z}, M)$ defined by the same diagrams. As k varies, this implies an isomorphism of simplicial groups $\mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M^\times) \simeq \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M)^\times$ where the monoid structure on the right is given by the one on M . Applying $Hom_{Mon\Delta^{op}PrShv(\mathcal{C})}(N, \)$ and using part (i), one deduces that
 $Hom_{Mon\Delta^{op}PrShv(\mathcal{C})}(N, \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M^\times))$
 $\simeq Hom_{Mon\Delta^{op}PrShv(\mathcal{C})}(N, \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M)^\times)$
 $\simeq Hom_{Mon\Delta^{op}PrShv(\mathcal{C})}(N, \mathbf{map}_{\Delta^{op}PrShv(\mathcal{C})}(\Delta^n, M))$ and the claim now follows by varying n , using the adjunction between \mathbf{map} and \times . \square

Using the above results, Theorem 2.1 now follows from the following chain of isomorphisms using the amplification of the indicated results provided by Lemma 2.6:

$$\begin{aligned} map_{MonSp^{\Sigma, T}(\mathcal{C})}(S[M], A) &\stackrel{2.3, iii)}{\simeq} map_{Mon\Delta^{op}PrShv(\mathcal{C})}(M, F \circ Ev_0(A)) \\ &\stackrel{2.9, ii)}{\simeq} map_{Groups\Delta^{op}PrShv(\mathcal{C})}(M, (F \circ Ev_0(A))^\times) \\ &\stackrel{2.9, i)}{\simeq} map_{Mon\Delta^{op}PrShv(\mathcal{C})}(M, \mathbf{map}_{Mon\Delta^{op}PrShv(\mathcal{C})}(\mathbf{Z}, F \circ Ev_0(A))) \\ &\stackrel{2.3, iii)}{\simeq} map_{Mon\Delta^{op}PrShv(\mathcal{C})}(M, \mathbf{map}_{MonSp^{\Sigma, T}(\mathcal{C})}(S[\mathbf{Z}], A)) \\ &= map_{Mon\Delta^{op}PrShv(\mathcal{C})}(M, \mathbf{G}'_m(A)). \end{aligned}$$

Recall that we may replace Mon by $AbMon$ everywhere. \square

Now, what happens if we try to give Theorem 2.1 a homotopy theoretic meaning, equipping everything with suitable model structures? One problem that may arise is the definition of the multiplicative group in Theorems

1.1 and 1.2 using the derived mapping spaces with respect to the chosen model structures. It is not clear if there is a cofibrant replacement of $S[\mathbf{Z}]$ which is also a comonoid, that is an affine derived *group* scheme. One may try to show that the functor represented by \mathbf{G}'_m is weakly equivalent to one factoring through simplicial abelian groups. Independently (in fact maybe not completely independently) of this, the main problem seems to be the following. Making our proof homotopy invariant means that all adjunctions involved have to be Quillen, and that will be impossible to achieve. The problem that appears does so already for the trivial category \mathcal{C} with a single object and no nontrivial automorphisms, that is for classical homotopy theory. Consider the adjunction of Lemma 2.3,(iii), restricted to *abelian* monoids. We want the model structure on $AbMon(\Delta^op Sets)$ to be the usual one. For Sp^Σ , we have essentially two families of model structures, namely the usual ones and the positive ones. If we choose a usual non-positive stable model structure, then this will not lift to a model structure on $AbMon(Sp^\Sigma)$ with weak equivalences and fibrations defined using the forgetful functor to Sp^Σ because of Lewis paradoxon, see e. g. [MMSS, section 14] or [SS1, Remark 4.5]. The fact that this adjunction is not Quillen for any reasonable model structure on abelian monoids is why we have to work so much more in the next two sections, using motivic versions of E_∞ -spectra, of the recognition principle, of a theorem of Schwede and Shipley [SS2] establishing a zig-zag of Quillen equivalences between $H\mathbf{Z}$ -modules in symmetric spectra and unbounded complexes of abelian groups, etc. This should not be considered as a technical problem about model category theory or symmetric spectra, but as an honest mathematical problem related to the stable homotopy type of the sphere spectrum and the content of our main theorem. Therefore it will appear in some way or another in any language one might choose to deal with these questions.

3 Model structures for algebras over operads in symmetric spectra and applications

The goal of this section is to show that the category of motivic symmetric spectra as considered by Jardine [Ja1] and Hovey [Ho2] equipped with suitable model structures satisfies all properties necessary for a motivic version of derived algebraic geometry. More precisely, we show that motivic symmetric spectra together with suitable model structures enable us to construct model structures for algebras in motivic symmetric spectra under a given operad, motivic symmetric spectra satisfy the assumptions of [TV2, section

1.1] when choosing suitable additional data (except that we do not discuss possible choices of \mathcal{C}_0 and \mathcal{A} as introduced in [TV2, 1.1.06 and 1.1.0.11] here). They also satisfy a motivic variant of the axioms of [GH, 1.1 and 1.4]. In particular, we construct model structures on E_∞ - and strictly commutative algebras over motivic symmetric spectra.

Later in this section, some further model structures and results related to simplicial presheaves and $H\mathbf{Z}$ -modules are considered as well.

3.1 Stable model structures

Let $\mathcal{C} = (Sm/S)_{Nis}$ and fix a cellular left proper model structure on $\Delta^{op}PrShv(\mathcal{C})$ which yields the Morel-Voevodsky [MV] unstable homotopy category $H(S)$, and similarly for the pointed variant $\Delta^{op}PrShv(\mathcal{C})_\bullet$. *Throughout this section, we will work with the motivic injective model structure of [MV] - or rather with its extension to simplicial presheaves as in [Ja1] - which Hirschhorn ([Hi1], see also [Hor, Corollary 1.6]) has shown to be cellular.* We denote the generating cofibrations (resp. trivial cofibrations) by I (resp. J). Besides being simplicial, cellular and proper, this model structure has two additional features which will be important in the sequel. First, the cofibrations are precisely the monomorphisms, in particular all objects are cofibrant. Second, it is a monoidal model category, that is it satisfies [Ho1, Definition 4.2.6]. This follows because smashing with any object preserves weak equivalences, compare [MV, Lemma 3.2.13] and [Hor, Theorem 1.9]. Note that the second condition of loc. cit. for being a monoidal model category is automatically satisfied because all objects are cofibrant.

We now fix an object T of $\Delta^{op}PrShv(\mathcal{C})_\bullet$. For many arguments below we may take an arbitrary T , but sometimes (e. g. in Theorem 3.6 and in Theorem 3.15) we will need that $T \simeq S^1 \wedge T'$ for a suitable T' , which holds in particular for $T = S^1$ and for $T = \mathbf{P}^1$. So we assume that $T \simeq S^1 \wedge T'$ for a suitable T' from now on, although many results do hold in greater generality.

We may apply [Ho2, Theorem A.9, Definition 8.7] and [Hi2, Theorem 4.1.1] to get a stable model structure on the category of motivic symmetric spectra $Sp^{\Sigma, T}(\mathcal{C})$ from the above unstable one on $\Delta^{op}PrShv(\mathcal{C})_\bullet$. This model structure coincides with the one of [Ja1, Theorem 4.15]. In particular, the *motivic stable equivalences* are those defined on p. 509 of loc. cit., that is defined with respect to injective stably fibrant objects.

Theorem 3.1 (*Hovey, Jardine*) *The above stable model structure on $Sp^{\Sigma, T}(\mathcal{C})$ is simplicial, proper, cellular and monoidal.*

Proof: By [Ja1, Theorem 4.15] we have a proper closed simplicial model structure. It remains to check the first condition of [Ho1, Definition 4.2.6] for a monoidal model category, that is the pushout-product axiom. For this we may either apply [Ho2, Theorem 8.11] as we have chosen a model structure on $\Delta^{op}PrShv(\mathcal{C})$ for which all objects are cofibrant, or directly quote [Ja1, Proposition 4.19]. \square

This stable model structure on spectra will be referred to as the *projective stable model structure*. The term “projective” refers to the way we obtained the stable structure from the unstable one, as the unstable model structure we started with really is an “injective” one. With respect to our fixed choice of the model structure on $\Delta^{op}PrShv(\mathcal{C})$, this is a motivic generalization of the model structure considered in [HSS, Theorem 3.4.4]. It will turn out that this model structure will not meet all our requirements, which is why we need to introduce a motivic version of the (positive) S -model=flat model structure of [HSS] and [Sh]. The reasons for considering flat and positive model structures will become clear below. In the approach of Toën-Vezzosi, the reason for considering the flat model structure is that a motivic generalization of [Sh, Corollary 4.3] provides a tool to reduce [TV2, Assumption 1.1.0.4 (2)] to [TV2, Assumption 1.1.0.3].

We will also need a *stable injective model structure* on motivic symmetric spectra, that is a model structure obtained by starting with the levelwise cofibrations and weak equivalences and then localize to obtain the stable model structure. This is necessary because some arguments below will use that the monomorphisms are cofibrations in a certain model structure, which means that for showing that a monomorphism $X \rightarrow Y$ is a weak equivalence it is sufficient to show that the quotient Y/X is contractible, that is weakly equivalent to a point. This model structure has been first considered by Jardine [Ja2].

Theorem 3.2 (*Jardine*) *There is a model structure on $Sp^{\Sigma, T}(\mathcal{C})$ with weak equivalences being the motivic stable equivalences and cofibrations being the levelwise monomorphisms. This model structure is simplicial and proper. It is called the injective stable model structure.*

Proof: See [Ja2, Theorem 10.5] except for right proper, which follows from the right properness of the stable projective model structure which has more fibrations and the same weak equivalences. \square

Next, we establish a flat stable and a positive flat stable model structure. The obvious identity morphisms between these four stable model structures,

that is injective, flat, positive flat and projective, are all Quillen equivalences of simplicial model categories. Of course, it is also possible to establish projective and injective positive stable model structures, but we will not need these.

As in the classical case, there is a functor from symmetric sequences in $\Delta^{op}PrShv(\mathcal{C})_\bullet$ to $Sp^{\Sigma, T}(\mathcal{C})$ which is left adjoint to the forgetful functor. We denote it by $T \otimes -$, and it enjoys the same formal properties as the functor $S \otimes -$ in [HSS]. See e. g. [Ho1, Definition 2.1.7] for the definition of $I - cof$ for a set of maps I in a category.

Definition 3.3 *A map is a motivic flat cofibration if it is in $T \otimes M - cof$ where M is the class of levelwise monomorphisms in symmetric sequences.*

As we already said above, we will define also define “positive” variants of the model structures (at least for the flat one below), following [MMSS, Definition 6.1, Definition 9.1 and p. 484] which will be necessary to define a model structure on strictly commutative symmetric ring spectra further below. This variant has fewer cofibrations than the non-positive (sometimes also called “absolute”) model structure. In particular, the motivic symmetric sphere spectrum $\Sigma_T^\infty S^0$ is no longer cofibrant, so the usual contradiction related to the “Lewis paradoxon” does not appear (see e.g. [MMSS, p. 484]). Indeed, if one does not work with the positive model structure, then in the notations of Theorem 3.17 below the condition (2) of [Hi2, Theorem 11.3.2] or equivalently [SS1, Lemma 2.3.(1)] that U takes relative LJ -cell complexes to stable weak equivalences will fail. Looking at the proofs for this condition (see in particular [Sh, Proposition 3.3] and [MMSS, Lemma 15.5]) one sees how the positive model structure arises. The key point seems to be that the argument in the proof of [MMSS, Lemma 15.5] starting with “Since Σ_i acts on $O(q)$ as a subgroup of $O(ni)$ ” (read “on Σ_q as a subgroup of Σ_{ni} ”) does not work if $n = 0$.

Theorem 3.4 *The category $Sp^{\Sigma, T}(\mathcal{C})$ admits a model structure with the weak equivalences being the stable motivic equivalences and cofibrations being the motivic flat cofibrations. This model structure is simplicial, monoidal and proper, and we call it the flat stable model structure. There is a positive variant which enjoys the same properties (including the pushout product axiom), except that the motivic sphere spectrum is not flat positive cofibrant.*

Proof: To establish the flat and the flat positive model structure, there are various possible proofs. The first one is to directly generalize the corresponding results of [Sh]. When adapting her definition of I' and J' to the

motivic case, one must work with our above sets I and J , of course. An alternative reference for [Sh, Proposition 1.2] which generalizes to the motivic case is [DK], see also [Re, Proposition 3.1.9]. The proof of [Sh, Proposition 1.3] goes through in the motivic case as well. Note that the model category on equivariant simplicial presheaves one obtains is left proper. (With a bit more work, one may also show that the proof of right properness of [HSS, Lemma 5.5.3 (2)] generalizes to the motivic case, using that the final argument carries over as the \mathbf{A}^1 -local model structure on $\Delta^{op}PrShv(\mathcal{C})$ is right proper by [Ja1, Theorem A.5], and that the proofs of [HSS, Theorem 3.1.14 and Lemma 3.4.15] do carry over. We don't need this here, but later in the proof of Theorem 3.17. Alternatively, one may deduce the motivic version of [Sh, Proposition 1.3] from Proposition 1.2 of loc.cit using Hirschhorn's [Hi2] or Smith's [Ba] generalization of Bousfield localization.

Still another approach is to simply quote [He, Theorem II.4.5] to obtain the global model structure for Σ_n -simplicial presheaves corresponding to [Sh, Proposition 1.3] and then impose cardinality bounds to see that this model structure is cofibrantly generated and even cellular. This is also done in [Hi1, Theorem 4.9] who attributes this result to Smith, and in [Ba, Theorem 2.16]. Hirschhorn or Smith localization then yields the \mathbf{A}^1 -local model structure on Σ_n -simplicial presheaves, with generating sets of (trivial) cofibrations different from the ones the approach of [Sh] yields. Propositions 2.1 of loc. cit is just the product model structure. When establish the motivic generalizations of Proposition 2.2 and Lemma 2.3 in [Sh], the arguments go through and yield a cofibrantly generated level model structure, which again is even cellular. To see this, observe that (both classical and) motivic symmetric spectra are cellular because simplicial presheaves are cellular, and so are products of cellular model categories. To check the three conditions of [Hi2, Definition 12.1.1] for (motivic) symmetric spectra, first observe that the third condition is [Ho2, Proposition A.4]. We then use the adjunction $(T \otimes -, U = Forget)$ between symmetric sequences and symmetric spectra, where U commutes with colims. The proof of the second condition is then similar to [Ho2, Lemma A.2]. Finally, for establishing the first property one proceeds as in the proof of [Ho2, Proposition A.8]. The argument there in fact slightly simplifies as we only have to consider one functor $T \otimes -$ rather than F_n for fixed n with intermediate considerations concerning F_m for other values of m . Still another alternative to this cellular approach is to use the theorem of Jeff Smith on the existence of left Bousfield localizations for combinatorial model categories, which has been written up recently by Barwick [Ba, Theorem 4.7]. (Still another variant would be to apply a more recent localization theorem of Bousfield as done in the appendix of [Sc2],

and this is certainly not the end of the list of variants of proofs...)

The level flat model structure on motivic symmetric spectra is left proper because the injective stable model structure which has more cofibrations and the same weak equivalences is left proper. To obtain the motivic version of [Sh, Theorem 2.4], that is passing from the level to the stable model structure, one applies Hirschhorn localization as in [Ho2, Definition 8.7]. (Depending on the variant of proof chosen above, one must choose a suitable cardinal number here to obtain a suitable set of morphisms in the class I .) It is also right proper because the stable projective model structure is right proper.

The proof that the positive model structure also exists again goes through in the motivic case. In more detail, the proof for the positive model structure is exactly the same, the only modification being that the motivic model structure generalizing the one of [Sh, Proposition 2.1] is defined as taking on Σ_0 -spaces the cofibrantly model structure with fibrations and weak equivalences being all morphisms. Then take the product model structure on motivic Σ_n -spaces for all $n \geq 0$ as before and proceed as in the non-positive case. As the positive model structure has fewer cofibrations, it is also left proper.

To check that these model structures are monoidal we must check the two conditions of [Ho1, Definition 4.2.6]. The second condition in the non-positive case is easy as the sphere spectrum is stably cofibrant because $* \rightarrow \mathrm{Spec}(S)_+$ is cofibrant in $\Delta^{op}PrShv(\mathcal{C})_\bullet$ and $T \otimes -$ is left Quillen. Hence it remains to check the first condition, that is the pushout product axiom. The proof of [HSS, Theorem 5.3.7] goes through, and may be even simplified a bit, see Lemma 3.5 below, which also applies to the positive variant. \square

Lemma 3.5 *Both the stable flat and the stable flat positive model structure on $Sp^{\Sigma, T}(\mathcal{C})$ satisfies the push-out product axiom.*

Proof: We only do the non-positive case. In the positive case, the condition on cofibrations follows similarly, and the one for stable equivalences then follows from the corresponding property for the non-positive model structure.

We start by observing that for any finite group G , the above model structure on G -objects in $\Delta^{op}PrShv(\mathcal{C})_\bullet$ is monoidal because $\Delta^{op}PrShv(\mathcal{C})_\bullet$ is monoidal and we have defined cofibrations and weak equivalences using the forgetful functor. It follows that the category of symmetric sequences in $\Delta^{op}PrShv(\mathcal{C})_\bullet$ is monoidal (see e. g. [Ha2, Theorem 12.2]). The stable flat

model structure on $Sp^{\Sigma, T}(\mathcal{C})$ has the same cofibrations as the level structure, so it remains to show that if given two cofibrations $f : A \rightarrow B$ and $g : X \rightarrow Y$ then if f (or g) is a stable equivalence then so is $f \wedge g : A \wedge Y \coprod_{A \wedge X} B \wedge X \rightarrow B \wedge Y$. This can be shown exactly as in [HSS, Theorem 5.3.7 (5)]. The argument goes through replacing as usual simplicial sets by simplicial presheaves and S^1 by T . In particular Lemma 3.1.6 of loc. cit. remains valid in this situation. \square

Note that similar to [HSS], the proof of this lemma provides a variant of the above proof that the stable projective model structure on $Sp^{\Sigma, T}(\mathcal{C})$ satisfies the pushout product axiom.

Next, we wish to study operads \mathcal{O} over motivic symmetric spectra. There are two approaches we are interested in: Simplicial operads, that is operads in simplicial sets for simplicial monoidal model categories, and internal operads in monoidal model categories. We sometimes apply one and sometimes the other point of view. Every operad in simplicial sets yields an internal operad in $Sp^{\Sigma, T}(\mathcal{C})$ via the monoidal functor Σ_T^∞ , and a similar argument applies to the unstable case of simplicial presheaves. The converse is not true, but all operads we are interested in are simplicial ones. We will establish a theorem on the existence of model structures for arbitrary internal operads and stable positive model structures (see Theorem 3.6), and weaker results in the non-positive case (see Proposition 3.9). The latter will be used when considering abjunctions of type (Σ_T^∞, Ev_0) for E_∞ -objects.

In general, one of the standard ways to construct a model structure on a category \mathcal{D} is to lift a cofibrantly generated model structure on a category \mathcal{C} along a right adjoint in a free/forgetful-style adjunction $\mathcal{C} \xrightleftharpoons{\quad} \mathcal{D}$, defining fibrations and weak equivalences in \mathcal{D} by applying the forgetful functor. If this does yield a model structure, then the adjunction is Quillen and we say that \mathcal{C} creates a model structure on \mathcal{D} . The main problem when checking the model axioms for \mathcal{D} is that in one of the factorizations obtained by the small object argument it is not clear that certain relative cell complexes are weak equivalences. See e.g. [SS1, Lemma 2.3], [Hi2, Theorem 11.3.2], [BM1, 2.5] and certain proofs below. One strategy for proving this is to establish a fibrant replacement functor, see e.g. the discussion in the remark after Proposition 3.11 below. Another strategy is to check the required property “by hand”. If one is unable to successfully apply one of these two strategies, one may - as first done by Hovey [Ho0] - try to proceed by weakening the axioms of a model category in a suitable way, which leads to the notion of a *semi-model category*. See e.g. [Fr, 12.1], we will not pursue this approach.

The following theorem is a generalization of a result of Harper [Ha1]. Compare also the article of Elmendorff and Mandell [EM] which establishes a similar result for simplicial operads.

Theorem 3.6 *Let \mathcal{O} be any operad in $Sp^{\Sigma, T}(\mathcal{C})$, and consider $Sp^{\Sigma, T}(\mathcal{C})$ with the positive flat stable model structure. Assume that $T \simeq S^1 \wedge T'$ for some pointed object T' . Then the forgetful functor from $\mathcal{O} - \text{alg}$ to $Sp^{\Sigma, T}(\mathcal{C})$ equipped with the stable flat positive model structure creates a model structure on $\mathcal{O} - \text{alg}$ in the sense of [SS1, Lemma 2.3].*

Proof: The proof is along the lines of [Ha1, Theorem 1.1]. We will indicate the nontrivial modifications to be made in the motivic setting. One proceeds by showing that the first condition of [SS1, Lemma 2.3] is satisfied. Using that transfinite compositions of acyclic monomorphisms are acyclic monomorphisms because they are the acyclic cofibrations for the model structure of Theorem 3.2, this boils down to show the motivic analog of [Ha1, Proposition 4.4]. All references here and below are with respect to the general set up of section 4 of loc. cit. which contains the proofs, but in the end we only care about symmetric sequences of symmetric spectra which are concentrated in degree zero, as discussed in section 7 of loc. cit. It is sufficient (notations taken from [Ha1]) to show that j_t is a weak equivalence for all t . We proceed as in Proposition 4.29 of loc. cit., using that there is a stable model structure on motivic spectra in which all monomorphisms are cofibrations. Hence we need motivic versions of Proposition 4.28 and 4.29 of loc. cit. The proof of Proposition 4.29 uses a five lemma argument which requires that smashing with S^1 detects stable weak equivalences, which is fine as $T \simeq T' \wedge S^1$. Everything else now carries over to the motivic case. (Note that the positive model structure is used in [Ha1, proof of Proposition 4.28] (“Since $m \geq 1...$ ”), and looking at his Calculation 6.15 one sees exactly what fails for $m = 0$.) \square

Note that the above forgetful functor admits the “free algebra” functor as a (Quillen) left adjoint, and also that both categories admit internal mapping spaces compatible under this adjunction. This will be generalized in Theorem 3.10 below.

We now shift our attention to non-positive model structures. The next conjecture is inspired by results of Harper and Schwede. It applies in particular to simplicial E_∞ -operads.

Conjecture 3.7 *Let \mathcal{O} be any simplicial operad such that the action of Σ_n on $\mathcal{O}(n)$ is free (by which we always mean objectwise and levelwise free away*

from the basepoint), and consider $Sp^{\Sigma, T}(\mathcal{C})$ with the absolute flat or pro-projective model structure. Then the forgetful functor from $\mathcal{O} - \text{alg}(Sp^{\Sigma, T}(\mathcal{C}))$ to $Sp^{\Sigma, T}(\mathcal{C})$ creates a model structure on $\mathcal{O} - \text{alg}(Sp^{\Sigma, T}(\mathcal{C}))$ in the sense of [SS1, Lemma 2.3].

There is the following strategy of proof, which is an attempt of a motivic generalization of a variant of a proof for classical symmetric spectra as sketched in [Sc2, section III.4], notations are again as in [Ha1]. In principle, it might be applicable to internal operads as well and not only to those with values in simplicial sets. As before, we are reduced to consider the push-out square of [Ha1, Proposition 4.4]. It is shown in [Ha2, Proposition 7.19] that the Σ_t -equivariant map $Q_{t-1}^t \rightarrow Y^{\otimes t}$ is an acyclic cofibration of symmetric spectra if $X \rightarrow Y$ is an acyclic cofibration. Using again the motivic generalization of [HSS, Theorem 5.3.7], this implies that the map $\mathcal{O}_A[t] \wedge Q_{t-1}^t \rightarrow \mathcal{O}_A[t] \wedge Y^{\otimes t}$ is a monomorphism and a weak equivalence for any \mathcal{O} -algebra A . Next, we show that the action of Σ_t on the motivic symmetric spectra $\mathcal{O}_A[t]$ is free. By definition, the action of Σ_t on $O_t(A)$ as defined after [Sc2, Remark 4.3] is free. Also by definition, $O_A[t]$ is an explicit coequalizer of $\mathcal{O}_t(A)$ with respect to two Σ_t -equivariant maps coming from another spectrum with free Σ_t -action. Now one has to check that the action of Σ_t on this quotient is also free. There are easy examples showing that this will not be true for arbitrary operads, so here one has to use the assumptions on the operad. Fresse even provided me with an example where the Σ_n -action on $\mathcal{O}(n)$ is free, but the action on $\mathcal{O}_n(A)$ is not. *This is the gap one has to fill when using this strategy of proof.* Hence the diagonal action of Σ_t on $O_A[t] \wedge Y^{\otimes t}$ is also free, and so are the ones on $O_A[t] \wedge Q_{t-1}^t$ and on $O_A[t] \wedge (Y^{\otimes t}/Q_{t-1}^t)$. That is, we have a cofiber sequence of Σ_t -free spaces

$$O_A[t] \wedge Q_{t-1}^t \rightarrow O_A[t] \wedge Y^{\otimes t} \rightarrow O_A[t] \wedge (Y^{\otimes t}/Q_{t-1}^t)$$

for the injective model structure, which remains true after dividing out the free Σ_t -action on all three objects. Indeed, taking coinvariants commutes with taking the cofiber as colimits commute among each other, and as we have a model structure in which the monomorphisms are the cofibration, the cofiber of $\mathcal{O}_A[t] \wedge Q_{t-1}^t \rightarrow \mathcal{O}_A[t] \wedge Y^{\otimes t}$ is also the homotopy cofiber. The fact that $\mathcal{O}_A[t] \wedge (Y^{\otimes t}/Q_{t-1}^t)$ is weakly equivalent to a point remains true after dividing out the action of Σ_t as the argument of [Sc2, Proposition III.4.12] applies to general simplicial monoidal model categories, including motivic symmetric spectra. Besides the gap above, this seems to be the only place where it might be an advantage to restrict to operads defined in simplicial sets rather than in motivic symmetric T-spectra). Hence $O_A[t] \wedge_{\Sigma_t} Q_{t-1}^t \rightarrow$

$O_A[t] \wedge_{\Sigma_t} Y^{\otimes t}$ is an acyclic cofibration for the stable injective model structure, and now the proof can be finished as the one of Theorem 3.6.

Note that the recent preprint [GG] contains a detailed discussion of techniques related to the problems above.

We can prove the above Conjecture 3.7 at least for the Barratt-Eccles operad, which will be sufficient for our purposes.

Definition 3.8 *Let \mathcal{W} be the Barratt-Eccles operad with values in the monoidal model category $(\Delta^{op}Sets, \times, pt)$, see e.g. [BeF]. It extends to an internal operad in $Sp^{\Sigma, T}(\mathcal{C})$ via the functor $(-)_+$ to pointed simplicial sets, the constant functor to $\Delta^{op}PrShv(\mathcal{C})_\bullet$ and Σ_T^∞ to $Sp^{\Sigma, T}(\mathcal{C})$ as all of these functors are monoidal. If we denote the internal operad in $Sp^{\Sigma, T}(\mathcal{C})$ by \mathcal{W} as well, the two notions of a \mathcal{W} -algebra in $Sp^{\Sigma, T}(\mathcal{C})$ thus obtained coincide by definition and the above adjunctions.*

Observe that in simplicial degree zero, \mathcal{W} is just the associative operad Ass , and a product of those in higher simplicial degrees.

We now prove the above conjecture for the operad \mathcal{W} . We don't know of a reference for this result even for classical symmetric spectra. The following proof also applies to other simplicial operads for which a decomposition pattern similar to the one below for \mathcal{W} may be established.

Proposition 3.9 *Conjecture 3.7 is true for $\mathcal{O} = \mathcal{W}$.*

Proof: According to the above strategy of proof, we must show that the coequalizer $\mathcal{W}_A[n]$ has a free action of Σ_n for all $n \geq 0$. By definition [Sc2, Construction III.4.8], [Ha1, Proposition 4.6], the motivic symmetric spectrum with Σ_n -action $\mathcal{W}_A[n]$ is the coequalizer of

$$(m, \mathcal{W}_n(\alpha)) : \mathcal{W}_n(\mathcal{W}(A)) \rightrightarrows \mathcal{W}_n(A)$$

where $\mathcal{W}_n(A) := \coprod_{p \geq 0} \mathcal{W}(n+p) \times_{\Sigma_p} A^p$ and $\mathcal{W}(A) := \mathcal{W}_0(A)$. The map $\alpha : \mathcal{W}(A) = \coprod_{p \geq 0} \mathcal{W}(p) \times_{\Sigma_p} A^p \rightarrow A$ is given by the \mathcal{W} -algebra structure of A . The map m is given by the operad structure of \mathcal{W} and will be described below, following [Sc2, Section III.4]. By definition, the freeness of the Σ_n -action for a symmetric spectrum has to be checked objectwise (if the site is non-trivial, that is in the motivic case), and then for every symmetric spectrum levelwise, and in each level degree-wise for the simplicial set. Also note that colimits in motivic symmetric spectra are constructed the same way (objectwise, levelwise, simplicially degree-wise), and so are products in

simplicial sets and more generally in simplicial presheaves, and similarly for the smash product in the pointed case. Finally, the smash product of a pointed simplicial set with a symmetric spectrum is given levelwise by the smash product of the pointed simplicial sets. All of this together implies that the whole argument really reduces to one of (pointed and even unpointed) simplicial sets, so we simplify our notation accordingly.

We now fix $n \geq 0$, and choose orbit decompositions of the sets $\Sigma_{n+m} = \mathcal{W}(n+m)_0$ with Σ_n -action by left multiplication for all $m \geq 0$. These decompositions yield decompositions of the simplicial sets $\mathcal{W}(n+m)$ for all $m \geq 0$, as $\mathcal{W}(n+m)$ in simplicial degree r is simply the $r+1$ -fold product of Σ_{n+m} with diagonal Σ_n -action. So we only spell out the decompositions in simplicial degree zero.

Recall that as a Σ_n -set with action given by left multiplication the set Σ_{n+1} decomposes as a coproduct of $n+1$ copies of Σ_n , and inductively Σ_{n+m} decomposes as a coproduct of $(n+m) \cdot \dots \cdot (n+1)$ copies of Σ_n for all $m \geq 0$. We now fix particular choices for these decompositions of all Σ_{n+m} for a fixed given n and all $m \geq 0$ once and for all, and consequently fix decompositions for all $\mathcal{W}(n+m)$.

In every simplicial degree $\mathcal{W}(n+m)$ is a finite set. The decompositions we choose may be written as products $\Sigma_{n+m} = \Sigma_n \times M_{m,n}$ where Σ_n acts by left multiplication on the left factor and trivially on the right factor. Our decomposition is then determined by the following. For any positive integer r , any element $\sigma \in \Sigma_r$ is uniquely determined by $\sigma(1, \dots, r)$. For $r = n+m$, we write $\sigma = \tau \times \rho$ with $\tau \in \Sigma_n$ being the element obtained by deleting all entries in $\sigma(1, \dots, r)$ which are larger than n , and $\rho \in M_{m,n}$ being determined by where we insert these remaining elements $n+1, n+2, \dots, n+m$ between the given permutation of $\tau(1, \dots, n)$.

It is obvious that $\mathcal{W}_n(\alpha)$ maps copies of Σ_n with respect to the above decomposition identically (that is not permuting the elements inside each copy of the Σ_n -set Σ_n) to copies of Σ_n . We will show that the same is true for the map m . Consequently, the coequalizer $\mathcal{W}_A[n]$ consists of free Σ_n -orbits as well, which finishes the proof.

According to loc. cit., the map $m : \coprod_{s \geq 0} \mathcal{W}(n+s) \times_{\Sigma_s} (\mathcal{W}(A))^{\times s} \rightarrow \coprod_{r \geq 0} \mathcal{W}(n+r) \times_{\Sigma_r} A^r$ is defined on the summand for a fixed $s \geq 0$ by the following composition of Σ_n -equivariant maps:

$$\begin{aligned} \mathcal{W}(n+s) \times_{\Sigma_s} \mathcal{W}(A)^{\times s} &\xrightarrow{\cong} \coprod \mathcal{W}(n+s) \times_{\Sigma_s} (\mathcal{W}(i_1) \times \dots \times \mathcal{W}(i_s) \times_{\Sigma_{i_1} \times \dots \times \Sigma_{i_s}} A^{\times i_1 + \dots + i_s}) \\ &\xrightarrow{\cong} \coprod \mathcal{W}(n+s) \times_{\Sigma_s} \mathcal{W}(1) \times \dots \times \mathcal{W}(1) \times \mathcal{W}(i_1) \times \dots \times \mathcal{W}(i_s) \times_{\Sigma_{i_1} \times \dots \times \Sigma_{i_s}} A^{\times i_1 + \dots + i_s} \end{aligned}$$

$$\begin{aligned}
&\rightarrow \coprod \mathcal{W}(n + i_1 + \dots + i_s) \times_{\Sigma_{i_1} \times \dots \times \Sigma_{i_s}} A^{\times i_1 + \dots + i_s} \\
&\rightarrow \coprod \mathcal{W}(n + r) \times_{\Sigma_r} A^{\times r}
\end{aligned}$$

where the last map is given by reindexing and the universal property of coproducts, and the second last map is given by the structure maps of the Barratt-Eccles operad \mathcal{W} . Now all four morphisms map free Σ_n -orbits identically to free Σ_n -orbits with respect to the Σ_n -decompositions introduced above. For the first, second and last morphism this is obvious. For the third map this can be checked using the equivariance condition of the operadic structure maps. To see this, again one first looks at the simplicial degree zero for which $\mathcal{W}(n + s)_0 = \Sigma_{n+s}$, that is the associative operad Ass . Then generalize to higher degrees as explained above, which involves cartesian products of Ass , and argue componentwise. \square

Once these results are established, one may deduce the motivic variant of [Ha1, Theorem 1.4]. Namely, we have the following.

Theorem 3.10 *Let $f : \mathcal{O} \rightarrow \mathcal{O}'$ be a morphism of operads and consider the lifts of the flat positive stable model structure on $Sp^{\Sigma, T}(\mathcal{C})$ to $\mathcal{O} - alg(Sp^{\Sigma, T}(\mathcal{C}))$ and $\mathcal{O}' - alg(Sp^{\Sigma, T}(\mathcal{C}))$. Then f induces an (enriched) Quillen adjunction*

$$f_* : \mathcal{O} - alg(Sp^{\Sigma, T}(\mathcal{C})) \rightleftarrows \mathcal{O}' - alg(Sp^{\Sigma, T}(\mathcal{C})) : f^*$$

which is a Quillen equivalence if f is a stable weak equivalence in every operadic degree.

Proof: By construction of our motivic model structures, the proof of [Ha1, Theorem 1.4] carries over. Enrichments are not mentioned in loc. cit., but the arguments given there immediately show they behave as well as expected. \square

In particular, the model categories of E_∞ -algebras and strictly commutative monoids in $Sp^{\Sigma, T}(\mathcal{C})$ are Quillen equivalent.

We continue to study the absolute=non-positive situation. If one can not prove Conjecture 3.7 for a given operad \mathcal{O} using the strategy discussed above, a different approach might be to first look at the level model structure as in the following result

Proposition 3.11 *There is a projective level model structure on $Sp^{\Sigma, T}(\mathcal{C})$ with fibrations and weak equivalences defined levelwise. This model structure lifts to $\mathcal{O} - alg(Sp^{\Sigma, T}(\mathcal{C}))$ for any operad \mathcal{O} .*

Proof: In the classical case (i.e. for the trivial site), the projective level model structure on $Sp^{\Sigma,T}(\mathcal{C})$ is introduced in [HSS] and the positive variant in [Sc2], [Sh] and [MMSS]. These level model structures are cofibrantly generated with respect to acyclic cofibrations I and J (resp. I^+ and J^+). To show that it lifts to $\mathcal{O} - alg(Sp^{\Sigma,T}(\mathcal{C}))$, as before the only thing that one has to check is that any map in $\mathcal{O}J$ -cell is a weak equivalence. For this one may proceed similarly to [EKMM, Lemma VII.5.6]. Namely, the geometric realizations of the maps in J are $(\Sigma^\infty \text{ of})$ inclusions of deformation retracts. Furthermore, these are stable under the free functor \mathcal{O} , under push-outs in $\mathcal{O} - alg(Sp^{\Sigma,T}(\mathcal{C}))$ (by refining an argument of [Ho1, Proposition 2.4.9], as Mandell kindly explained to me) and under sequential colimits. Note that geometric realization does preserve colimits, and a map in $Sp^{\Sigma,T}(\mathcal{C})$ is a level equivalence if and only if its geometric realization is.

In the motivic case (that is for the non-trivial site), the argument has to be refined a bit. Looking at diagram categories, one obtains global level (absolute and positive) model structures on $Sp^{\Sigma,T}(\mathcal{C})$ with J consisting of inclusions of deformation retracts (objectwise, in the classical sense) and consequently on $\mathcal{O} - alg(Sp^{\Sigma,T}(\mathcal{C}))$. To obtain the \mathbf{A}^1 -local level model structures, one applies Bousfield-Hirschhorn localization to $Sp^{\Sigma,T}(\mathcal{C})$ and to $\mathcal{O} - alg(Sp^{\Sigma,T}(\mathcal{C}))$ to a suitable set S , which precisely yields exactly the cofibrantly generated motivic level model structure of [Ho2, Theorem 8.2] on $Sp^{\Sigma,T}(\mathcal{C})$. For $\mathcal{O} - alg(Sp^{\Sigma,T}(\mathcal{C}))$ one applies Bousfield-Hirschhorn localization with respect to the set obtained by applying the free functor \mathcal{O} to S . \square

Note that if we could apply Bousfield-Hirschhorn or Bousfield-Smith localization with respect to a suitable set of stable weak equivalences, this would yield an alternative proof of Conjecture 3.7. The problem both with Hirschhorn's and Smith's approach is that $\mathcal{O} - alg(Sp^{\Sigma,T}(\mathcal{C}))$ with respect to the level model structure is not left proper in general, as one sees looking e. g. at $\mathcal{O} = Comm$. We will not pursue this approach in our article.

Remark: There is a model structure on the category of operads in simplicial sets given by [BM1, Theorem 3.2 and Example 3.3.1], in particular weak equivalences and fibrations are defined on the underlying simplicial sets. Note that this model structure is different from the one of [Re]. Looking at [BM1, 4.6.4] for classical symmetric spectra, the argument in the proof of [BM1, Theorem 3.5.(a)] applies to simplicial model categories in general, defining \mathcal{E}_f in the category of simplicial sets and applying SM7. In particular, we may apply this theorem to a fibrant replacement map $f : X \rightarrow X_{fib}$ in $Sp^{\Sigma,T}(\mathcal{C})$ and a cofibrant model P for the E_∞ -operad with respect to

the model structure of [BM1, Example 3.3.1]. Consequently, if we can show (which we can't) that the hypothesis in Theorem 3.5 (a) of loc. cit. holds, namely that $f^{\wedge n}$ is a trivial cofibration for all $n > 0$, then we have constructed a fibrant replacement for any object in $\mathcal{O} - \text{alg}(Sp^{\Sigma, T}(\mathcal{C}))$. Note however that this construction is not functorial. Hence it is not clear if we may apply a variant of an argument of Quillen [Qu], see e. g. [Re, Proposition 3.1.5] or [Sc1, B.2 and B.3], to lift the model structure of $Sp^{\Sigma, T}(\mathcal{C})$ to a model structure on $\mathcal{O} - \text{alg}(Sp^{\Sigma, T}(\mathcal{C}))$. In general, in a monoidal model category in which all objects are cofibrant, the hypothesis holds as can be shown by an easy induction. More generally, to check the hypothesis in a monoidal model category (a similar argument then presumably applies to simplicial model categories and simplicial instead of internal operads), [BM1, Remark 3.6] claims it is enough to have a set of generating trivial cofibrations having cofibrant domains. Classical symmetric spectra with the (absolute=non-positive) flat or projective model structure satisfy this property and are monoidal model categories. But it is not clear why this helps as Harper gave an easy example of a cofibrantly generated monoidal model category whose generating acyclic cofibrations have cofibrant domains, but where the above property for $f^{\wedge n}$ fails already for $n = 2$. In short, the techniques of [BM1] do not provide an absolute stable model structure for \mathcal{O} -algebras in symmetric spectra with \mathcal{O} a simplicial or internal cofibrant operad. \square

3.2 Unstable model structures

We will now establish an unstable variant of Theorem 3.6, as well as Quillen adjunctions between unstable and stable model categories. We also show that a motivic generalization of the axioms of Goerss-Hopkins holds. For the sake of completeness, we recall that there is also an unstable projective model structure on simplicial presheaves, starting with fibrations and weak equivalences defined objectwise and then localizing with respect to the Nisnevich topology and the affine line as in the unstable injective case above. The identity functor obviously induces Quillen equivalences between the global (and hence between the local) unstable projective and injective model structure. Note that the unstable local projective model structure is also simplicial, cellular and monoidal [Bl], [Hor]. We will only use the unstable injective model structure in the sequel.

Theorem 3.12 *For any operad \mathcal{O} , the unstable injective motivic model structures on simplicial presheaves lifts via the forgetful functor to the category $\mathcal{O} - \text{alg}(\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet})$ of \mathcal{O} -algebras in this model category.*

Proof: We have a motivic fibrant replacement functor in $\mathcal{O} - \text{alg}(\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet})$ as the usual fibrant replacement functor commutes with products, so the argument of Quillen discussed above applies, see also [Re, Proposition 3.2.5]. The existence of the motivic \mathbf{A}^1 -local fibrant replacement functor on $\mathcal{O} - \text{alg}(\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet})$ follows from the existence of a fibrant replacement functor on $\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet}$ which commutes with \times . More precisely, both motivic fibrant replacement functors constructed in [MV, Lemma 2.3.20 and Lemma 3.2.6] commute with finite limits by [MV, Theorem 2.1.66 and p. 97]. (Note that for the special case $\mathcal{O} = \mathcal{W}$, we may of course alternatively adapt the above proof for $Sp^{\Sigma, T}(\mathcal{C})$.) \square

Theorem 3.13 *There is a Quillen adjunction*

$$\Sigma_T^{\infty} : \Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet} \xrightleftharpoons{\quad} Sp^{\Sigma, T}(\mathcal{C}) : Ev_0$$

where both functors are strong monoidal. Here $\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet}$ is equipped with the above unstable injective local model structure, and $Sp^{\Sigma, T}(\mathcal{C})$ is equipped with the projective, flat or injective stable model structure built from it as discussed above.

Proof: The adjunction was already established in the first section. The functor Σ_T^{∞} preserves weak equivalences. This can be seen using that the suspension functor to ordinary motivic spectra does and that the forgetful functor detects stable motivic equivalences [Ja1, Proposition 4.8]. It also preserves cofibrations by a motivic generalization of [Sc2, Proposition III.1.5] with respect to the stable projective model structure, and hence also with respect to the stable flat and injective model structure. \square

The following result is why we had to establish a non-positive model structure on $\mathcal{O} - \text{alg}(Sp^{\Sigma, T}(\mathcal{C}))$.

Corollary 3.14 *Let \mathcal{O} be an operad such that the forgetful functor to $\mathcal{O} - \text{alg}(\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet})$ creates an absolute stable projective resp. flat model structure on the latter, e.g. $\mathcal{O} = \mathcal{W}$. Then the Quillen adjunction of Theorem 3.13 induces a Quillen adjunction*

$$\Sigma_T^{\infty} : \mathcal{O} - \text{alg}(\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet}) \xrightleftharpoons{\quad} \mathcal{O} - \text{alg}(Sp^{\Sigma, T}(\mathcal{C})) : Ev_0$$

for the stable projective resp. flat model structures on $\mathcal{O} - \text{alg}(\Delta^{\text{op}} \text{PrShv}(\mathcal{C})_{\bullet})$.

Proof: This follows immediately from Theorem 3.13 and the definition of the model structures on \mathcal{O} -algebras via forgetful functors. \square

We now show that some of the above model structures feed into the motivic version of the obstruction theory machine for E_∞ ring spectra of Goerss and Hopkins [GH]. The machinery of Toën and Vezzosi [TV2] will be discussed further below. As we already said in the introduction, this result has been obtained independently by Ostvaer.

Theorem 3.15 *Both the flat and the projective stable positive model structures on $Sp^{\Sigma,T}(\mathcal{C})$ satisfy the motivic analog of the five axioms in [GH, 1.1 and 1.4].*

Proof: Everything has been shown above already except that the generating cofibrations and the generating acyclic cofibrations can be chosen to have cofibrant source and condition (5). As every object in $\Delta^{op}PrShv(\mathcal{C})_\bullet$ is cofibrant, the sources of the generating cofibrations and of the generating acyclic cofibrations for the stable flat model structure on $Sp^{\Sigma,T}(\mathcal{C})$ are cofibrant because $T \otimes -$ is left Quillen, hence this holds in particular for the stable flat positive model structure. Consequently, the same is true for the model structure of operads as the proof of Theorem 3.6 shows that the forgetful functor $\mathcal{O} - alg \rightarrow Sp^{\Sigma,T}(\mathcal{C})$ is right Quillen. To show the claim for the generating acyclic cofibrations, note that essentially the same argument goes through for those, as the domains of the motivic variant of the class K of [HSS, Definition 4.3.9] also have cofibrant sources because Ev_n is right Quillen for the flat level model structure, hence F_n preserves cofibrations, and the model structures satisfy the pushout product axiom. Alternatively, we may simply quote [Hi2, Proposition 4.5.1]. Condition (5) follows again from the motivic variants of Harper’s results discussed above. Namely, one first uses [Ha1, Proposition 4.28 (a)] applied to $* = Spec(k) \rightarrow X$ and then applies the motivic variant of [Ha1, Proposition 4.29(b)]. \square

3.3 HA-contexts

We now establish model structures for commutative ring spectra and algebras over those and establish the properties required in the axioms of [TV2]. For this, we consider the category $Comm(Sp^{\Sigma,T}(\mathcal{C}))$ of commutative unital monoids in $Sp^{\Sigma,T}(\mathcal{C})$. The notation $Comm$ is taken from [TV2], further below we write $AbMon$ instead which is more consistent with unstable notations. The forgetful functor $U : Comm(Sp^{\Sigma,T}(\mathcal{C})) \rightarrow Sp^{\Sigma,T}(\mathcal{C})$ has a

left adjoint L , namely the obvious motivic variant of [Sh, section 3]. For $R \in \text{Comm}(Sp^{\Sigma,T}(\mathcal{C}))$, we define the category $R\text{-mod}$ in the usual way.

Theorem 3.16 *Consider the stable flat absolute or positive model structure on $Sp^{\Sigma,T}(\mathcal{C})$, and let R be an arbitrary object in $\text{Comm}(Sp^{\Sigma,T}(\mathcal{C}))$. Then there is a model structure on $R\text{-Mod}(Sp^{\Sigma,T}(\mathcal{C}))$ where the weak equivalences and fibrations are defined using the forgetful functor $R\text{-Mod}(Sp^{\Sigma,T}(\mathcal{C})) \rightarrow Sp^{\Sigma,T}(\mathcal{C})$ where the latter is equipped with the (absolute or positive) flat model structure. These model structures are monoidal, proper and combinatorial. Moreover, we have a Quillen adjunctions*

$$R \wedge - : (Sp^{\Sigma,T}(\mathcal{C})) \xrightleftharpoons{\rightarrow} R\text{-Mod}(Sp^{\Sigma,T}(\mathcal{C})) : U.$$

and

$$U : R\text{-Mod}(Sp^{\Sigma,T}(\mathcal{C})) \xrightleftharpoons{\rightarrow} Sp^{\Sigma,T}(\mathcal{C}) : \text{Map}(R, -)$$

with respect to the flat model structure.

Proof: The existence of the model structures follow from the model structures of Theorem 3.4 by applying either Kan's lifting theorem [Hi2, Theorem 11.3.2] using $R \wedge -$ as left adjoint, or the essentially equivalent [SS1, Theorem 4.1 (2)]. We don't know if the monoid axiom holds (compare [Ho2, p. 107]), but it is sufficient to check the second condition of [Hi2, Theorem 11.3.2] (or equivalently the first condition of [SS1, Lemma 2.3]), the first one is obvious. If R is cofibrant, then as the stable flat model structure is monoidal, the claim follows as explained in [SS1, Remark 4.2]. In fact, the monoid axiom probably holds in our case by some variant of this argument, but we won't need this.

For arbitrary R one must use the motivic generalization of [HSS, Theorem 5.3.7 (5)], compare the classical proof of [Sc2, Theorem IV.1.4]. Note that [TV2, Assumption 1.1.0.2] requires the model structure also for non-cofibrant R . Both left and right properness in $R\text{-mod}$ follow from left and right properness of $Sp^{\Sigma,T}(\mathcal{C})$. Right properness is immediate as the weak equivalences and fibrations in $R\text{-mod}$ are defined by the forgetful functor to the left proper model category $Sp^{\Sigma,T}(\mathcal{C})$. To show left properness, one uses that the generating cofibrations are level monomorphisms, hence so are all relative cell complexes built from them. Now use [Ho2, Corollary 2.1.15].

That the two adjunctions exists follows in the standard way. That the first adjunction is Quillen follows from the definition of the model structures involved. To prove that the second adjunction is Quillen, one must simply show that for cofibrant R the functor $\text{Map}(R, -)$ preserves fibrations and

trivial fibrations, which immediately follows from the fact that $Sp^{\Sigma,T}(\mathcal{C})$ is a monoidal model category. \square

It seems possible to prove the above result for the projective variant as well, but we won't need this.

Remark: One must show that the above model structure is combinatorial as desired by Toen and Vezzosi. For this, observe that simplicial sets are small [Ho1, Lemma 3.1.1], hence so are diagram categories over it (see also [SS1, Remark 2.4]). See [HSS, Proposition 3.2.13] for how to use this to show that symmetric spectra (and thus motivic symmetric spectra using a similar argument) are also small. So instead of choosing quite explicit sets of generating (trivial) cofibrations for the above model structures, one might like to take instead all cofibrations resp. trivial cofibrations with codomains bounded by α (a cardinal that in a suitable sense is large enough with respect to $Sp^{\Sigma,T}(\mathcal{C})$) as the set of generating (trivial) cofibrations. Then it remains to check that the (trivial) fibrations are indeed those of the model structure, that is it is enough to check the lifting property of a (trivial) on these sets. In most examples this is not hard to see. As all model categories we have studied are cofibrantly generated, smallness implies that they are all combinatorial.

For $R \in Comm(Sp^{\Sigma,T}(\mathcal{C}))$, we denote the category of commutative R -algebras by $R - Comm(Sp^{\Sigma,T}(\mathcal{C}))$. We have a forgetful functor $U : Comm(Sp^{\Sigma,T}(\mathcal{C})) \rightarrow Sp^{\Sigma,T}(\mathcal{C})$.

Theorem 3.17 *The stable flat positive model structure on $Sp^{\Sigma,T}(\mathcal{C})$ creates a proper combinatorial model structure on $Comm(Sp^{\Sigma,T}(\mathcal{C}))$ where f is a weak equivalence (resp. fibration) if and only if Uf is. If $R \in Comm(Sp^{\Sigma,T}(\mathcal{C}))$, then the same is true for $R - Comm(Sp^{\Sigma,T}(\mathcal{C}))$.*

Proof: The existence of the cofibrantly generated model structure on $Comm(Sp^{\Sigma,T}(\mathcal{C}))$ follows from Theorem 3.6 applied to the operad $Comm$.

(In particular, this provides an alternative to the proof of [Sh] which relies on [Hi2, Theorem 11.3.2], see also [Sc2, Theorem A.1.4], [SS1, Lemma 2.3] and [MMSS, Proposition 5.13]. Note also that [Sc2] assumes that U commutes with filtered colimits which implies that all small colimits exist, which is an assumption in [Hi2]. As all objects are small, the only nontrivial thing of the assumptions that is left for Shipley to check is that LJ -cell complexes - recall that J are the generating trivial cofibrations - are stable weak equivalences, which is not so easy and relies on Propositions 3.3 and 3.4 of her article.)

As discussed above, it is easy to see that the underlying category is locally presentable, and thus is combinatorial.

By the same formal argument as the one in the proof of [Sh, Theorem 3.2], the model structure on $R - \text{Comm}(Sp^{\Sigma, T}(\mathcal{C}))$ follows from the one of $\text{Comm}(Sp^{\Sigma, T}(\mathcal{C}))$. It remains to show properness. The model structures on $\text{Comm}(Sp^{\Sigma, T}(\mathcal{C}))$ and more generally on $R - \text{Comm}(Sp^{\Sigma, T}(\mathcal{C}))$ are right proper by the argument of [Sh, proof of Proposition 4.7] as we can prove the motivic generalization of the variant of [HSS, Lemma 5.5.3 (2)] for positive level fibrations, compare the discussion in the proof of Theorem 3.4. To check that the model structure is also left proper, one uses the motivic analogue of [HSS, Corollary 5.3.10] - which follows from the motivic generalization of [HSS, Theorem 5.3.7] and Ken Brown's lemma - and then proceeds as in the proof of [Sh, Proposition 4.7]. \square

Moreover, we have the following, which yields the axioms 1.1.0.3 and 1.1.0.4(2) in the definition of a HA -context as considered by Toen and Vezzosi [TV2]. Observe that axiom 1.1.0.3 is not a formal consequence of the “monoidal” established in Theorem 3.16.

Proposition 3.18 *Consider the stable flat positive model structure on $Sp^{\Sigma, T}(\mathcal{C})$, which we already have shown to be monoidal.*

- (i) *The monoidal model structure on $Sp^{\Sigma, T}(\mathcal{C})$ is symmetric monoidal.*
- (ii) *Let $R \in \text{Comm}(Sp^{\Sigma, T}(\mathcal{C}))$. Then for any cofibrant $M \in R - \text{mod}$, the functor $- \wedge_R M$ preserves weak equivalences. For any cofibrant $B \in R - \text{Comm}(Sp^{\Sigma, T}(\mathcal{C}))$, the functor $B \wedge_R - : R - \text{mod} \rightarrow B - \text{mod}$ preserves weak equivalences.*

Proof: Part (i) is clear. Part (ii) follows from a motivic generalization of the ideas of [Sh, section 4] which can be carried out thanks to the results we already established. More precisely, observe that using the motivic generalization of [Sh, Corollary 4.3] (which holds as [Sh, Proposition 4.1] generalizes to the motivic situation), Assumption 1.1.0.4 (2) reduces to Assumption 1.1.0.3, which is a motivic generalization of [HSS, Lemma 5.4.4]. This motivic generalization holds as we have already observed that [HSS, Theorem 5.3.7, Corollary 5.3.10] generalize to the motivic case. \square

The above results establish [TV2, Assumptions 1.1.0.1 - 1.1.0.4] for motivic symmetric spectra $Sp^{\Sigma, T}(\mathcal{C})$. The non-unital variant (take away the index 0 in the definition of the free functor L) mentioned in [TV2, 1.1.0.4.(1)] (compare also Remark 1.1.0.5 of loc. cit.) follows again from Theorem 3.6 applied to the reduced commutative operad Comm_{nu} with

$(Comm_{nu})_0 = \Sigma_T^\infty(pt)$ when considered as an internal operad in $Sp^{\Sigma,T}(\mathcal{C})$. For classical symmetric spectra, this was already stated in [TV2, Example (4) following Remark 1.1.0.7].

Remark: In [TV2, p. 20 example (4)], it is stated that [TV2, Assumptions 1.1.0.1 - 1.1.0.4] hold for symmetric spectra (that is \mathcal{C} being the trivial category in our setting) by the results of [Sh], but some details of why this is so are not explained in full detail. Most of this is carried out in the arguments above. It remains to show that all categories involved are locally presentable. The standard references for locally presentable categories are [AR] and [Bor]. I do not know a reference for a detailed proof why symmetric spectra are locally presentable, but this easily follows from the smallness property as explained above. The reason for this condition is that [TV2] quote unpublished work from J. Smith - the relevant parts are now available thanks to [Ba] - in order to ensure that certain localized model structures exist, see e. g. [TV2, section 1.3.1]. Our localization arguments in the first half rely on the published work of [Hi2] on cellular model categories instead, but the arguments of Smith do apply just as well. So it is rather a matter of personal taste if one works with Smith's or with Hirschhorn's version of Bousfield localization.

Note that Assumption 1.1.0.4 (2) (which is part of Proposition 3.18 (ii)) is probably not true for the positive projective model structure but only for the positive flat model structure (compare also [MMSS, Theorem 14.5]). Here is why [Sh, Proposition 4.1] (which is an ingredient of the proof of [Sh, Corollary 4.3]) fails for the projective model structure already for $R = S$. In the notation of loc. cit., the proof uses that the maps of $\mathbf{P}_S(S^+I)$ are S -cofs, which is deduced from the fact that the maps in $\mathbf{P}(I^{l+})$ are coproducts of monomorphisms of symmetric sequences. It is not clear that the corresponding maps $\mathbf{P}_S(S^+I)$ are stable cofibrations for the projective model structure.

3.4 $H\mathbf{Z}$ -modules

Theorem 3.16 will be an ingredient in the proof of the Main Theorems, but will only be used in the case where $T = S^1$ and $R = H\mathbf{Z}$ is the usual simplicial Eilenberg Mac Lane spectrum considered as objectwise constant simplicial presheaf. Such an object R is called flat resp. projective if it is cofibrant in $Sp^{\Sigma,T}(\mathcal{C})$ with respect to the flat resp. projective stable model structure. So if you don't care about [TV2], the following Lemma allows you to take a short-cut.

Lemma 3.19 *Both the classical and the objectwise constant motivic S^1 -spectrum $H\mathbf{Z}$ are flat.*

Proof: It suffices to show that $H\mathbf{Z}$ is flat as a classical symmetric spectrum. Using the adjunction between constant presheaves and presheaves, it follows that the presheaf of S^1 -spectra $H\mathbf{Z}$ is cofibrant for the global flat stable model structure as well, and further (motivic left) localizations do not change the cofibrations. But as a classical symmetric spectrum, $H\mathbf{Z}$ is flat cofibrant, see [Sc2]. \square

It remains to lift the model structures on “naive” $H\mathbf{Z}$ -modules and $Ch(\mathbf{Ab})$ to motivic (=Nisnevich- \mathbf{A}^1 -local) model structures on presheaves of those as well, using similar techniques as before. For this we first recall the relevant classical model structures.

Theorem 3.20 *The category $Ch(\mathbf{Ab})$ has a model structure with weak equivalences being the quasi-isomorphisms and fibrations the epimorphisms. The full subcategory $Ch(\mathbf{Ab})_{\geq 0}$ has a model structure with weak equivalences being the quasi-isomorphisms and fibrations the epimorphisms in degree ≥ 1 . The inclusion $incl$ and the good truncation $\tau_{\geq 0}$ form a Quillen adjunction between these model categories. The category $\Delta^{op}\mathbf{Ab}$ has a model structure with weak equivalences and fibrations being the weak equivalences and fibrations of the underlying simplicial sets. The Dold-Kan correspondence between $\Delta^{op}\mathbf{Ab}$ and $Ch(\mathbf{Ab})_{\geq 0}$ is an isomorphism of model categories. All three model categories are cofibrantly generated and left proper.*

Proof: For the cofibrantly generated model structures, see [Qu] for $Ch(\mathbf{Ab})_{\geq 0}$, [Ho1, Theorem 2.3.11] for $Ch(\mathbf{Ab})$ and [Ho1, Theorem III.2.8 and Theorem III.2.12] for $\Delta^{op}\mathbf{Ab}$ (or use the lifting argument from [SS1, Lemma 2.3.(2)] as before for the latter, recalling that there is a fibrant replacement functor in $\Delta^{op}Sets$ which preserves products). The claims about the Quillen adjunction resp. equivalence are now straightforward. Left properness for $Ch(\mathbf{Ab})$, and hence for the other two model categories as well, follows from [Ho1, Proposition 3.2.9]. \square

Theorem 3.21 *Theorem 3.20 generalizes to the corresponding motivic model categories.*

Proof: Use the same techniques as before. First, pass to diagram categories, that is presheaves with values in the above model categories, and then

Bousfield-Hirschhorn-Smith-localize with respect to the Nisnevich topology and to the affine line. Note that $Ch(\mathbf{Ab})$ is cellular by [Ho1, Lemma 2.3.2].

□

We will need one more auxiliary model category, namely “naive” $H\mathbf{Z}$ -modules, taken from another article of Schwede and Shipley [SS2, Definition B.1.1]. Again, we first explain the classical case. Once more using the above techniques, everything generalizes to the motivic case using [Hi2, Proposition 12.1.5 and Theorem 13.1.14] and Bousfield-Hirschhorn localization. We omit the details.

Definition 3.22 *A naive $H\mathbf{Z}$ -module is a collection of pointed simplicial sets $\{M_n\}_{n \geq 0}$ and associative and unital action maps $(H\mathbf{Z})_p \wedge M_q \rightarrow M_{p+q}$. A morphism of naive $H\mathbf{Z}$ -modules is a map of graded pointed simplicial sets which is strictly compatible with the action of $H\mathbf{Z}$.*

As shown in [SS2, Theorem B.1.3], the category $NvH\mathbf{Z} - mod$ of naive $H\mathbf{Z}$ -modules has a model structure in which the fibrations and the weak equivalences are created by the forgetful functor $U : NvH\mathbf{Z} - mod \rightarrow Sp$ from naive $H\mathbf{Z}$ -modules to classical Bousfield-Friedlander spectra with the standard stable model structure of [BF, Theorem 2.3]. The model structure on $NvH\mathbf{Z} - mod$ is cellular and left proper, the latter by the same argument as in the proof of Theorem 3.16. One may also consider adjoints to U as in the case of symmetric spectra (compare Theorem 3.16), but we won’t need this in the sequel.

Theorem 3.23 (Schwede-Shipley) *There is a zig-zag of Quillen equivalences*

$$H\mathbf{Z} - mod \xleftarrow{\sim} NvH\mathbf{Z} - mod \xrightarrow{\sim} Ch(\mathbf{Ab}).$$

Proof: See [SS2, Appendix B].

□

4 Proof of the main theorems

Having established all necessary model structures and Quillen adjunctions in the previous section, we are now ready to prove the Main Theorems 1.1 and 1.2.

Let \mathcal{M} be a simplicial monoidal model category and T be a suitable pointed object in \mathcal{M} . We are mostly interested in two cases. Theorem 1.1 is about $\mathcal{M} = \Delta^op Sets$ and $T = S^1$. Theorem 1.2 is about \mathcal{M} being

simplicial presheaves on Sm/S with the \mathbf{A}^1 -local model structure recalled at the beginning of section 3, and $T = \mathbf{P}^1$, although throughout the proof motivic symmetric spectra over $T = S^1$ will be considered as well. Note that the statements of the theorems are independent of the model structure one chooses, and by the previous section there is at least one for all the categories involved. Furthermore, we wish to emphasize once more that much of the proofs here, and in fact those of section 3 as well, generalizes to other (simplicial) monoidal model categories. On the other hand, there are some key results which do not generalize. In particular, the Theorems of [SS2] quoted below, and also Theorem 4.2 and Theorem 4.4 are really fundamental results specifically about classical Eilenberg Mac Lane spectra and delooping along the classical circle S^1 , respectively. Results without \mathcal{M} made explicit hold in the very general situation described above. We fix $N \in AbMon(\mathcal{M})$ and $A \in AbMonSp^{\Sigma, T}(\mathcal{M})$, and assume that N is group-like, as defined below.

Putting everything together, we obtain the following diagram of categories and adjunctions, with the left adjoint displayed on top as usual. For simplicity, we only exhibit this diagram in the classical version, that is for $\mathcal{M} = \Delta^{op}Sets$. It generalizes to diagram categories, in particular with the site $\mathcal{C} = (Sm/S)_{Nis}$ as index category, and to various motivic localizations of those, as explained above and below. All categories are simplicial model categories, and all adjunctions are Quillen (one is even an actual equivalence of categories preserving the model structure). The global picture is that there are compatible forgetful functors U from the left to the right column. The functor V in the right column is defined in [HSS, 4.3] and extends [SS2, B.1] to a functor L in the left column. The functor U in the top row is studied in Theorem 3.16. The upper right adjunction is a Quillen equivalence [HSS, Theorem 4.2.5] for $\mathcal{M} = \Delta^{op}Sets$ (here Sp denotes Bousfield-Friedlander spectra [BF]), and by [Ja1, Theorem 4.40] for motivic symmetric S^1 -spectra. The left column is explained at the end of the previous section. The dotted adjunction between $E_\infty - alg(\Delta^{op}Sets)$ and Sp is standard in other models for the stable homotopy category, see Lemma 4.1 below. The other dotted arrow to which it restricts is an equivalence of homotopy categories enriched over $Ho(\Delta^{op}Sets)$ as proved in [ABGHR, Theorem 3.45], compare our Theorem 4.2 below. This is a variant of the famous *recognition principle* due to May [Ma] and Boardman-Vogt [BV]. We write E_∞ for the Barratt-Eccles operad \mathcal{W} introduced in Definition 3.8. The precise meaning of the dotted arrows in our setting will be explained further below.

$$\begin{array}{ccc}
HZ - mod & \xrightarrow{U} & Sp^\Sigma \\
\downarrow U \simeq L & & \downarrow U \simeq V \\
NvHZ - mod & \xrightarrow{U} & Sp \\
\uparrow \simeq \mathcal{H} & & \uparrow \tau_{\geq 0} \\
Ch(\mathbf{Z} - mod) & & Sp_{\geq 0} \\
\downarrow \tau_{\geq 0} \quad \uparrow incl & & \uparrow \simeq \text{recogn.pr} \\
Ch(\mathbf{Z} - mod)_{\geq 0} & & \\
\downarrow \cong & & \\
\Delta^{op} \mathbf{Ab} & \xrightarrow{U} & E_\infty - alg(\Delta^{op} Sets)_{grl} \xrightleftharpoons[GL_1]{incl} E_\infty - alg(\Delta^{op} Sets)
\end{array}$$

(A dashed arrow points from Sp to $E_\infty - alg(\Delta^{op} Sets)$)

Lemma 4.1 *In the world of Lewis-May-Steinberger spectra [LMS], we have a Quillen adjunction of topological model categories*

$$\Sigma^f : E_\infty - alg(\Delta^{op} Sets) \xrightleftharpoons{\quad} Sp : \Omega^f$$

enriched over topological spaces.

Proof: See [ABGHR, Lemma 3.43]. □

The construction of the functor Ω^f uses the linear isometries operad which is built in the definition of Lewis-May-Steinberger spectra, and thus is a key ingredient when proving the following theorem in their setting.

Theorem 4.2 *There is an equivalence of homotopy categories*

$$Ho(E_\infty - alg(\Delta^{op} Sets)_{grl}) \simeq Ho(Sp_{\geq 0}).$$

enriched over $Ho(\Delta^{op} Sets)$.

We will prove this theorem in our situation, that is for Bousfield-Friedlander spectra in simplicial sets Sp and its motivic generalizations. For this we will use a variant of Lemma 4.1 which we learned from Schwede, and the symmetric spectra variant of which will presumably be included in the final version of [Sc2]. Namely, we consider the following zig-zag diagram of Quillen adjunctions

$$Sp \xrightleftharpoons{\quad} Sp(E_\infty - alg(\Delta^{op} Sets)) \xrightleftharpoons{\quad} E_\infty - alg(\Delta^{op} Sets)$$

with left adjunctions displayed on top and $Sp(E_\infty - alg(\Delta^{op}Sets))$ being the category of spectra with spaces and spectral structure maps all being E_∞ . The left free/forgetful adjunction creates a model structure on $Sp(E_\infty - alg(\Delta^{op}Sets))$ as usual, that is using the same arguments as for the existence of the stable model structure on Sp from $\Delta^{op}Sets$. (This also can be done in the motivic case below, starting with the model structure on $(E_\infty - alg(\Delta^{op}PrShv(\mathcal{C})))$ established in Theorem 3.12.) In the right Quillen adjunction, the functor $Ev_0 : Sp(E_\infty - alg(\Delta^{op}Sets)) \rightarrow E_\infty - alg(\Delta^{op}Sets)$ is the usual evaluation at the 0th space which is a right Quillen functor. However its left adjoint is *not* the naive Σ^∞ , but defined using the simplicial model structure on $E_\infty - alg(\Delta^{op}Sets)$ (which is induced by the Quillen adjunction with $\Delta^{op}Sets$) when defining the smash products with S^n to define the level n -space of an object in $Sp(E_\infty - alg(\Delta^{op}Sets))$.

Now by the recognition principle, the left Quillen adjunction induces an equivalence of (enriched) homotopy categories, and hence (see e. g. [Ho1, Proposition 1.3.13]) is an (enriched) Quillen equivalence. Moreover, the right Quillen adjunction induces an equivalence of (enriched) homotopy categories $Ho(E_\infty - alg(\Delta^{op}Sets)_{grl}) \simeq Ho(Sp(E_\infty - alg(\Delta^{op}Sets))_{\geq 0})$. The latter equivalence can also be formulated as a Quillen equivalence, using suitable localizations of the above model structures on $E_\infty - alg(\Delta^{op}Sets)$ and $Sp(E_\infty - alg(\Delta^{op}Sets))$ as we now explain. (We do explain the localized model structure on Sp only, for $Sp(E_\infty - alg(\Delta^{op}Sets))$, the arguments are exactly the same.) In modern language, see e.g. Hirschhorn [Hi2, section 5] or Smith [Ba, section 5], these are examples of *right* Bousfield localizations, that is increasing the number of weak equivalences while keeping the same fibrations. I do not know of any published reference for the following proposition for sequential or symmetric spectra, but I learned that there is work in progress by Sagave and Schlichtkrull - now available, see [SaS] - who apply similar techniques to study similar questions for I -spaces. One should also compare [Pe, section 3.2] for a detailed discussion of how to lift the motivic Postnikov decomposition to the level of model structures using right Bousfield localizations, which contains precisely the arguments needed in our slightly easier case. Of course, the set C_{eff}^0 of loc cit. simply becomes the set of S^1 -suspension spectra of the simplicial spheres.

Proposition 4.3 (i) *The category Sp has a simplicial model structure with the same fibrations as the ones in [BF, Theorem 2.3] and weak equivalences the π_n -isomorphisms for $n \geq 0$.*

(ii) *The category $E_\infty - alg(\Delta^{op}Sets)$ has a simplicial model structure with the same fibrations as above, that is fibrations on underlying simplicial*

sets, and a map being a weak equivalence if it is one after restricting to the invertible components.

Proof: We apply the dual of Bousfield's theorem [Bou, Theorems 9.3 and 9.7] to the Postnikov truncation $Q = \tau_{\geq 0}$ on Sp resp. to $Q = (-)^\times =$ unital components on $E_\infty - alg(\Delta^{op}Sets)$ and the in both cases obvious transformation α . The category Sp is proper by [BF, Theorem 2.3], and the properties (A1) and (A2) of [Bou, 9.2] are obviously satisfied. The dual of axiom (A3) follows as the hypothesis of loc. cit. yield a homotopy pushout square square and hence a long exact sequence of homotopy groups. The category $E_\infty - alg(\Delta^{op}Sets)$ is right proper because the forgetful functor to the proper model category $\Delta^{op}Sets$ preserves limits, fibrations and weak equivalences. Showing that it is also left proper is a bit more subtle, see [Sp, Theorem 4] (or [Fr, Theorem 12.4.B]). The definition of left proper in loc. cit. coincides with the usual definition as all objects in $\Delta^{op}Sets$ are cofibrant. In order to apply the theorem of loc. cit. concerning left properness, we need to now that the operad in question is cofibrant for the model structure of loc. cit., which is created by the one of symmetric sequences. That one is equipped with the product model structure of equivariant simplicial sets which in turn is created by the one of $\Delta^{op}Sets$ forgetting the group action. This implies that the Barratt-Eccles-operad \mathcal{W} is Σ -cofibrant, which by definition means that its underlying symmetric sequence is cofibrant, as all Σ_n act freely. Now we choose a cofibrant replacement \mathcal{W}_{cof} of \mathcal{W} (which itself is not cofibrant as Fresse kindly explained to me), which then in particular is also Σ -cofibrant (see e. g. [BM1, Proposition 4.3]). Finally, the model categories $\mathcal{W} - alg$ and $\mathcal{W}_{cof} - alg$ are Quillen equivalent by [Fr, Theorem 12.5.A], so we do not distinguish between them in our notations in the sequel. The conditions (A1), (A2) and (A3) of [Bou] are again easy to check. \square

Putting everything together and again suppressing the Quillen equivalence between Sp and $Sp(E_\infty - alg(\Delta^{op}Sets))$ in our notations, we obtain the following square of Quillen adjunctions. It corresponds to the lower right triangle in the large diagram of Quillen adjunctions above before Lemma 4.1.

$$\begin{array}{ccc}
 E_\infty - alg(\Delta^{op}Sets) & \begin{array}{c} \xrightarrow{Id} \\ \xleftarrow{Id} \end{array} & E_\infty - alg(\Delta^{op}Sets) \\
 \begin{array}{c} \uparrow \text{Ev}_0 \\ \downarrow H \end{array} & & \begin{array}{c} \uparrow \text{Ev}_0 \\ \downarrow H \end{array} \\
 Sp & \begin{array}{c} \xrightarrow{Id} \\ \xleftarrow{Id} \end{array} & Sp
 \end{array}$$

We claim that the left vertical pair is a Quillen equivalence. The hor-

horizontal equivalences are the right Bousfield localizations we just described. In more detail, the cofibrant objects in the left hand side model categories are the cofibrant objects with respect to the model structures on the right hand side which are moreover group-like E_∞ -spaces resp. (-1) -connected spectra. Hence the Quillen adjunction on the right hand side induces one on the left hand side. This is an example of [Hi2, Theorem 3.3.20(2)(a)]: note that [Hi2, Definition 8.5.11 (2)(a)] applied to right localization with respect to $\tau_{\geq 0}$ does not produce additional weak equivalences in E_∞ -algebras and therefore induces a Quillen adjunction between the lower left and the upper right corner in the above diagram. Composing this Quillen adjunction with the one on top leads to the one of the left hand side we are looking for. As we already pointed out above, this Quillen adjunction then induces an equivalence of homotopy categories by the recognition principle, hence it is a Quillen equivalence (see e. g. [Ho1, Proposition 1.3.13]).

Recall that when writing $E_\infty - \text{alg}(\Delta^{op}\text{Sets})_{grl}$ resp. $Sp_{\geq 0}$ in the large diagram above and further below, we really mean the model categories $E_\infty - \text{alg}(\Delta^{op}\text{Sets})$ resp. Sp with the right localized model structures established in Proposition 4.3. The total right derived functors of the right adjoint identity functors in the above square are precisely GL_1 resp the Postnikov functor $\tau_{\geq 0}$, thus justifying the labels on the arrows in the big diagram further above. This finishes our discussion of the proof of Theorem 4.2 and its refined formulation in the language of model categories. \square

Again, all above Quillen adjunctions and equivalences in the above diagram generalize to the motivic situation using always the same kind of arguments involving left Bousfield localizations of diagram categories.

Theorem 4.4 *The above Quillen adjunctions induce Quillen adjunctions for the corresponding motivic categories, which then induce an equivalence of homotopy categories*

$$Ho(E_\infty - \text{alg}(\Delta^{op}\text{PrShv}(\mathcal{C}))_{grl}) \simeq Ho(Sp^{S^1}(\mathcal{C})_{\geq 0})$$

enriched over $Ho(\Delta^{op}\text{Sets})$ and even over $Ho(\Delta^{op}\text{PrShv}(\mathcal{C}))$.

Proof: The above Quillen adjunction generalizes to one between global model structures on diagram categories (see e. g. [Hi2, Theorems 11.6.1 and 11.6.5]). We claim that this one then induces a Quillen adjunction after left Bousfield localization on both sides with respect to the Nisnevich topology and to $\mathbf{A}^1 = \mathbf{A}_S^1 \rightarrow S$ by standard arguments, that is applying [Hi2,

Theorem 3.3.20 (1)(a)]. On $(E_\infty - \text{alg}(\Delta^{op} \text{PrShv}(\mathcal{C})))$, this is precisely the model structure established in Theorem 3.12. Indeed, looking at the fibrant replacement functors discussed there, we see that we obtain the correct \mathbf{LFC} in the notation of loc. cit.. We then wish to apply the dual of [Bou, Theorem 9.3] to these left localized motivic model structures, thus obtaining the homotopy categories of connected motivic S^1 -spectra and grouplike motivic E_∞ -spaces using the (diagram versions of) the right Bousfield localizations considered in Proposition 4.3. To see that these right localizations of the \mathbf{A}^1 -local structures on motivic S^1 -spectra and on motivic E_∞ -spaces exist, we may apply the dual of [Bou] to them as before. Motivic S^1 -spectra are cellular and proper by Hovey and Jardine, that is (the sequential spectra version of) Theorem 3.1. The arguments above imply that motivic E_∞ -spaces are also cellular and left proper. To show that it is right proper follows as motivic spaces are right proper and the \mathbf{A}^1 -local model structure on E_∞ -spaces is created by the forgetful functor which preserves pull-backs. Finally, one checks the dual of the remaining hypotheses of [Bou]. By [Hi2, Theorem 3.3.20(2)(a)], we thus obtain a Quillen adjunction between the model categories corresponding to the homotopy categories appearing in the theorem we wish to establish. To see that this Quillen adjunction between these right localized motivic model structures is even a Quillen equivalence as claimed, use [Hi2, Theorem 3.3.20(1)(b)] applied to the Nisnevich and \mathbf{A}^1 -localizations on the diagram categories equipped with the right localized global model structures. \square

As Peleaz explained to me, Morel's connectivity result (which is valid only for $S = \text{Spec}(k)$) really is stronger than what we have used here. It can of course not be recovered using only the above techniques.

Now we are ready for the proof of the main theorems. We write down a chain of natural weak equivalences of simplicial sets, so applying π_0 yields Theorems 1.1 and 1.2. To simplify notation, we drop all base points in the sequel.

Let \mathcal{M} be as above. We have

$$\begin{aligned} & R\text{map}_{\text{AbMon}(Sp^{\Sigma, T}(\mathcal{M}))}(\Sigma_T^\infty N, A) \\ & \simeq R\text{map}_{E_\infty(Sp^{\Sigma, T}(\mathcal{M}))}(\Sigma_T^\infty N, A) \end{aligned}$$

using Theorem 3.10 and flat positive stable model structures. Now the identity is a Quillen equivalence between the positive and the non-positive (see Theorem 3.6 and Proposition 3.9) model structure, so we may switch to the latter on $E_\infty(Sp^{\Sigma, T}(\mathcal{M}))$. Then using Corollary 3.14, we have

$$\simeq R\text{map}_{E_\infty(\mathcal{M})}(N, Ev_0(A))$$

As N is an abelian group by assumption, hence grouplike, we have

$$\simeq Rmap_{E_\infty(\mathcal{M})_{grl}}(N, GL_1(Ev_0(A)))$$

by Proposition 4.3 above. Recall the meaning of the heuristic notations $E_\infty(\mathcal{M})_{grl}$ and GL_1 as introduced immediately after loc. cit., it would be more accurate to say that “ N is cofibrant in the right localized model structure of Proposition 4.3”. The chain of weak equivalences continues with

$$\simeq Rmap_{Sp_{\geq 0}}(HN, gl_1(A))$$

using Theorem 4.2 and its proof, which defines HN and $gl_1(A)$. Observe that $GL_1 \circ Ev_0 = Ev_0 \circ gl_1$, and moreover gl_1 allows a model-theoretic description as right Quillen adjoint similar to GL_1 . As before, we have suppressed the left hand side Quillen equivalence in the zig-zag of the Quillen adjunctions after Theorem 4.2 from our notations. Recall also that we are dealing with S^1 -spectra and the usual Eilenberg-Mac Lane spaces here. The next weak equivalence

$$\simeq Rmap_{NvH\mathbf{Z}-Mod(Sp_{\geq 0}(\mathcal{M}))}(HN, Rmap_{Sp_{\geq 0}(\mathcal{M})}(H\mathbf{Z}, gl_1(A)))$$

is just a formal adjunction, see Theorem 3.16 which allows a variant for “naive” $H\mathbf{Z}$ -modules. Note in particular that HN is a module over $H\mathbf{Z}$.

$$\simeq Rmap_{Ab(\mathcal{M})}(N, Rmap_{E_\infty(\mathcal{M})_{grl}}(\mathbf{Z}, GL_1(Ev_0(A))))$$

using Theorem 3.23 of Schwede-Shipley resp. its motivic generalization and Theorems 3.20 and 3.21. Note that this is compatible with Theorem 4.2 by the large commutative diagram above. In particular, $Rmap_{E_\infty(\mathcal{M})_{grl}}(\mathbf{Z}, GL_1(Ev_0(A)))$ no longer denotes a (presheaf of) $H\mathbf{Z}$ -module(s), but the corresponding (presheaf of) simplicial abelian group(s). Observe that the model structures on $Ab(\mathcal{M})$ and $AbMon(\mathcal{M})$ are compatible since both are created via the forgetful functor to \mathcal{M} .

$$\simeq Rmap_{AbMon(\mathcal{M})}(N, Rmap_{E_\infty(\mathcal{M})}(\mathbf{Z}, GL_1(Ev_0(A))))$$

$$\simeq Rmap_{AbMon(\mathcal{M})}(N, Rmap_{E_\infty(\mathcal{M})}(\mathbf{Z}, Ev_0(A)))$$

as \mathbf{Z} is grouplike again by Proposition 4.3 above

$$\simeq Rmap_{AbMon(\mathcal{M})}(N, Rmap_{E_\infty(Sp^{\Sigma, T}(\mathcal{M}))}(\Sigma_T^\infty \mathbf{Z}, A))$$

and finally proceeding as above

$$\simeq Rmap_{AbMon(\mathcal{M})}(N, Rmap_{AbMon(Sp^{\Sigma, T}(\mathcal{M}))}(\Sigma_T^\infty \mathbf{Z}, A)).$$

Note that the above chain of weak equivalences really arises from “zig-zags”, as various of the enriched Quillen equivalences in the above argument go in the “wrong” direction. To start with, the identity is a left Quillen adjoint from the positive to the absolute model structure on symmetric spectra and algebras over those. When considering derived mapping spaces, we must choose a cofibrant and a fibrant replacement functor for both model structures. In this situation, we may simply choose the cofibrant replacement functor with respect to positive model structure (which then also is one for

the absolute model structure) and the fibrant replacement functor with respect to the absolute model structure (which then is also one for the positive model structure). These choices show that the chain of weak equivalences leading to the Main Theorems really can be chosen to be one which is natural both in N and A . Another such zig-zag is hidden in the recognition principle, and still another one in the motivic generalization of [SS2].

Putting everything together, we therefore have a natural weak equivalence of derived simplicial mapping spaces

$$\begin{aligned} & Rmap_{AbMon(Sp^{\Sigma,T}(\mathcal{M}))}(\Sigma_T^\infty N, A) \\ & \simeq Rmap_{AbMon(\mathcal{M})}(N, Rmap_{AbMon(Sp^{\Sigma,T}(\mathcal{M}))}(\Sigma_T^\infty \mathbf{Z}, A)) \end{aligned}$$

which after applying π_0 finishes the proof of the main theorems.

5 Motivic preorientations and orientations of the derived multiplicative group

In this section, we explain how the Theorems 1.1 and 1.2 lead to the results about orientations and K -theory stated in the introduction.

Theorem 1.2 applies to $N = \overline{W}(GL_1) \simeq \mathbf{P}^\infty \in \Delta^{op}PrShv(\mathcal{C})$. Here, $\overline{W}(\cdot)$ is a specific model for the classifying space of a simplicial group, see [GJ, Chapter V, 4]. It is easy to see that $\overline{W}(\cdot)$ sends commutative simplicial abelian groups to commutative monoids in $\Delta^{op}PrShv(\mathcal{C})$. The equivalence $\overline{W}(GL_1) \simeq \mathbf{P}^\infty$ is a special case of [MV, Proposition 3.7]. Beware of the difference between \mathbf{G}_m and GL_1 . In Theorem 1.1, the same argument applies to the topological group $S^1 = U(1)$ with classifying space \mathbf{CP}^∞ .

We do not suggest a definition of the notion of a derived group scheme in $AbMonSp^{\Sigma,T}(\mathcal{C})$ here. Compare [Lu1, section 3] for a motivation of the following definition, at least for the trivial site.

Definition 5.1 (i) A pre-orientation on a derived group scheme G over A is an element in $Hom_{AbMon\Delta^{op}PrShv(\mathcal{C})}(\overline{W}(GL_1), G(A))$.

Note that this definition is related, but not equivalent to more classical notions of orientations as e. g. in Adams book [Ad]. However, it is the “correct” definition in order to obtain the right definition of tmf , and in the height 1 case to obtain KO .

One easily checks that for any simplicial abelian monoid, the associated suspension spectrum is a commutative motivic ring spectrum (compare [Sc2,

Example I.2.32]). Theorem 1.2 may thus be rephrased as follows in the special case of \mathbf{P}^∞ , where we write $S[-]$ for $\Sigma_T^\infty(-)$ following Lurie's notation (as introduced in the beginning of section 2 already).

Theorem 5.2 *There is a bijection between preorientations of \mathbf{G}_m over A and $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{AbMon}Sp^{\Sigma,T}(\mathcal{C}))}(S[\overline{W}(GL_1)], A)$. In other words, $S[\overline{W}(GL_1)]$ classifies preorientations of the derived multiplicative group.*

Lurie also gives a definition of an orientation, see [Lu1]. We do not suggest a motivic generalization of this definition in general, either. However, any reasonable generalization of the notion of an orientation from the classical to the motivic case will certainly imply a bijection between the set of orientations on \mathbf{G}_m over A and the set $\mathrm{Hom}_{\mathrm{Ho}(\mathrm{AbMon}Sp^{\Sigma,T}(\mathcal{C}))}(S[\overline{W}(GL_1)][\beta^{-1}], A)$ for a certain lift of the motivic Bott element β (see below). In other words, $S[\overline{W}(GL_1)][\beta^{-1}]$ will classify orientations of the derived multiplicative group.

By recent work of Spitzweck-Ostvaer [SO] and independently of Gepner-Snaith [GS], we have the following algebraic version of Snaith's theorem.

Theorem 5.3 (*Spitzweck-Ostvaer, Gepner-Snaith*) *There is an isomorphism of commutative monoids in $SH(S)$ between the underlying motivic spectrum of $S[\overline{W}(GL_1)][\beta^{-1}]$ and Voevodsky's motivic spectrum representing algebraic K -theory [Vo1, section 6.2] where $\beta \in \pi_{2,1}(BGL_1)$ is a lift of the motivic Bott element.*

In light of this result, our Theorem 5.2 above may be rephrased by saying that *algebraic K -theory classifies orientations of the derived multiplicative group*. More precisely, one must either assume S regular here or work with Weibel's homotopy invariant algebraic K -theory [We] for non-regular base schemes.

The classical Snaith theorem [Sn] together with some considerations about suspension spectra and suitable localizations of those being semistable symmetric ring spectra leads to a description of topological K -theory as a strictly commutative ring spectra, that is an abelian monoid in symmetric spectra. More precisely, suspension spectra are “semistable” in the sense of [Sc2, Theorem I.4.42] by [Sc2, Example I.4.46]. It follows that $S[\overline{W}(GL_1)][\beta^{-1}]$ is again a symmetric spectrum by [Sc2, Corollary I.4.67], hence complex topological K -theory is represented by a strictly commutative ring spectrum in Sp^Σ . This argument generalizes to the motivic situation.

Proposition 5.4 (*Röndigs, Spitzweck, Ostvaer*) *The object $S[\overline{W}(GL_1)][\beta^{-1}]$ is a commutative monoid in $Sp^{\Sigma, T}(\mathcal{C})$.*

Proof: See [RSO]. □

Returning again to the classical case, the fact that $K_{top}^{h\mathbf{Z}/2} \simeq KO_{top}$ implies that KO classifies all oriented derived multiplicative groups, see [Lu1, Remark 3.12] for details. A similar statement for algebraic and hermitian K -theory, namely $K_{alg}^{h\mathbf{Z}/2} \simeq KO_{alg}$ was conjectured to hold for arbitrary rings with 2 invertible at least after a suitable completion, see [Wi, 3.4.2]. This conjecture has been proved in many cases, see [Ko], [BKO] and more recently [HKO], [Sch], but in general it is wrong as [BKO] show.

Finally, let us mention that there are of course many examples of abelian monoids in symmetric T -spectra. Suspension spectra of abelian monoids, e.g. of algebraic groups or abelian varieties, are obvious examples. Another example is Voevodsky's algebraic cobordism spectrum **MGL**, as explained in [PY, section 6.5], [PPR, section 2.1]. The techniques of Schlichtkrull [Sk] then yield many more examples, as the proof of Theorem 1.1 of loc. cit. carries over to the motivic Thom spectrum, and hence applies to \mathcal{TU}/BGL with \mathcal{U} being the category of motivic spaces, that is simplicial presheaves. Moreover, one may try to use the isomorphism $\mathbf{A}^n - 0 \simeq S^{n-1} \wedge \mathbf{G}_m^n$ to extend the picture to a motivic version of generalized Thom spectra with respect to a motivic version of BF , that is with self-maps on T^n . We might pursue this topic in some other article.

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